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On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems. I. General Case

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In the context of dissipative and accumulative differential equations (which contain the spectral parameter λ nonlinearly) in a separable Hilbert space $\mathcal{H}$ we introduce a characteristic operator $M(\lambda)$ that works as an analog of the characteristic Weyl–Titchmarsh matrix. Its existence and properties are investigated. A description of $M(\lambda)$ that corresponds to separated boundary conditions is given. Analogs for Weyl functions and solutions are introduced. Weyl type inequalities for those analogs are established, which reduce to well-known inequalities in various special cases. The proofs are based on description and properties of maximal semi-definite subspaces in $\mathcal{H}^2$ of special form that we provide while studying boundary problems for equations as above.

Key words: operator differential equation, characteristic operator, characteristic projection, solution of Weyl type, maximal semi-definite subspace.
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This work shares the subjects of [1] and considers the operator differential equation in a separable Hilbert space $\mathcal{H}$ as follows:

$$\frac{i}{2}\left((Q(t)x(t))' + Q(t)x'(t)\right) - H_\lambda(t)x(t) = w_\lambda(t)f(t), t \in \bar{\mathcal{I}}, \mathcal{I} = (a,b) \subseteq R^1, \quad (0.1)$$

with $Q(t) = Q^*(t)$, $Q^{-1}(t)$, $H_\lambda(t) \in B(\mathcal{H})$; $Q(t) \in AC_{loc}$; the operator-valued function $H_\lambda(t) = H^*_{\bar{\lambda}}(t)$ being Nevanlinna; the weight $w_\lambda(t) = ImH_\lambda(t)/Im\lambda \geq 0$ ($Im\lambda \neq 0$) (see a more detailed description of the properties of $H_\lambda(t)$, $w_\lambda(t)$ in Sect. 1, the notions of AC_{loc} are treated in the sense of [2, 3]). Since the weight

$w_\lambda(t)$ can be degenerate, these considerations cover many types of dissipative and accumulative differential and difference equations, as it is demonstrated in [4, 5].

In Section 1 a characteristic operator (c.o.) for the equation (0.1) is introduced, its existence (Th. 1.2) and properties (Th. 1.1) are established. In this setting, the results of [4, 6–11] are covered (a more detailed exposition of those works is presented in [1]). In Section 3 we present necessary and sufficient conditions (Th. 3.1) for c.o. correspond to separated boundary conditions, which reduce with constant $Q(t)$ to conditions established in [1].* Those conditions are in claiming that c.o. admits a special expression via a projection, which is called characteristic. Under constant $Q(t)$, a part of these results has been announced in [1] without proofs.

In Section 4 we introduce operator analogs for Weyl functions and solutions of (0.1); those are used to describe the characteristic prodjections. Also for those analogs the Weyl type inequalities are established, which reduce in various special cases to the well-known inequalities [13–17]. Note that the defining properties of Weyl solutions for the scalar Sturm–Liouville equation and the two-dimensional Dirac system have been established by V.A. Marchenko [16]. The Weyl function for abstract operators has been introduced and investigated in [18] (see also [19, 20]).

The proof of a great deal of our results involve the linear manifolds of the form

$$\mathcal{L} = \{A_1 f \oplus A_2 f | f \in D\} \subset \mathcal{H}^2, \tag{0.2}$$

where A_j, $j = 1, 2$, are linear operators in $\mathcal{H}$, $D = D_{A_1} = D_{A_2}$. These linear manifolds are semi-definite in an indefinite metric generated by an operator of the form $Q = diag(Q_1, -Q_2)$, with $Q_j = Q_j^* \in B(\mathcal{H})$, $Q_j^{-1} \in B(\mathcal{H}), j = 1, 2$. In particular, the Q-nonnegativity for $\mathcal{L}$ means that

$$(Q_1 A_1 f, A_1 f) - (Q_2 A_2 f, A_2 f) \geq 0, \quad \forall f \in D. \tag{0.3}$$

In Section 2 we obtain a description for all such pairs of linear operators A_1, A_2 that the linear manifold of the form $\mathcal{L}$ (0.2) is maximal Q-semi-definite. (In the latter case such pair with $\dim \mathcal{H} < \infty$, $Q_1 = Q_2 = J = J^{-1}$ is called J-nonstretching or J-stretching, nonsingular pair in terms of the J-theory by V.P. Potapov. A different description for such pair was found by S.A. Orlov [21] using the J-theory). As a consequence we obtain a description of maximal Q-semi-definite linear manifolds $\mathcal{L}$ of the form (0.2) in terms of the linear condition that makes related the vectors $A_1 f$, $A_2 f$.

Note that (as it has been demonstrated in this work) the maximal Q-semi-definite linear manifold $\mathcal{L}$ of the form (0.2) reduces to either a dissipative or accumulative relation (of a special form) in $\mathcal{H}$. In the general case a description

*A corrector's notice: also expounded in [12].

of such relation is given in terms of its Cayley transform in [22] (see also [2, 3]). On the contrary, our approach is based on describing maximal definite subspaces in the space of M.G. Krein (see [23–25]).

In Section 2 we also produce necessary and sufficient conditions (Ths. 2.3, 2.7) for the inequality (0.3) being separated, that is equivalent to the following two inequalities being satisfied simultaneously:

$$(Q_1 A_1 f, A_1 f) \geq 0, \quad (Q_2 A_2 f, A_2 f) \leq 0.$$

It is just Th. 5.7 where generally unbounded idempotent (specific for $\dim \mathcal{H} = \infty$) arises. In the case of $\mathcal{L}$ (0.2) corresponding to the equation (0.1) it becomes a characteristic projection from $B(\mathcal{H})$.

Also in Section 2 we consider the case when in (0.2) the operators $A_j = A_j(\lambda)$, $j = 1, 2$, depend analytically on λ in a domain $\Lambda \subset C$. In this case we obtain a condition under which inequality (0.3) being separated for some $\lambda = \mu_0 \in \Lambda$ implies its separation for all $\lambda \in \Lambda$.

We denote the scalar products and norms in various spaces by $(\cdot, \cdot)$ and $\|\cdot\|$ (along with distinguishing indices if necessary). In view of its large volume, the work splits into three parts. Part I is formed by Sect. 1, Part II — by Sect. 2, and Part III — by Sects. 3, 4.

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1. Characteristic Operator. Its Definition, Properties, and Existence

Throughout the text we assume that for the operator-valued function $H_\lambda(t) = H^*_{\bar\lambda}(t) \in B(\mathcal{H})$ in (0.1) there exists on $\mathcal{I}$ a conull set ε such that:

1) There is such set $\mathcal{A} \supseteq C \backslash R^1$ that every its point has a neighborhood independent of $t \in \varepsilon$, in which $H_\lambda(t)$ depends analytically on λ at every fixed $t \in \varepsilon$.

2) For all $\lambda \in \mathcal{A}$, $H_\lambda(t)$ is Bochner locally integrable in the uniform operator topology (B-integrable).

3) The weight $w_\lambda(t) = Im H_\lambda(t)/Im\lambda$ is nonnegative ($t \in \varepsilon$, $Im\lambda \neq 0$). (Use the fact that $H_\lambda(t)$ is Nevanlinna to show that for all $\mu \in \mathcal{A} \bigcap R^1$, $t \in \varepsilon$ $\exists u - \lim\limits_{\lambda \to \mu \pm i0} w_\lambda(t) = w_\mu(t)$, with the operator-valued function $w_\mu(t)$ being locally B-integrable).

Let $X_\lambda(t)$ be the operator solution of (0.1) with $f(t) = 0$, normalized by the condition $X_\lambda(c) = I$, where $c \in \bar{\mathcal{I}}$, I denotes the identity operator in $\mathcal{H}$ (in terms of [24] $X_\lambda(t)$ is the Cauchy operator of (0.1)). Introduce the notation $Q(c) = G$.

It follows from $H_\lambda(t) = H^*_{\bar\lambda}(t)$ that with $\lambda \in \mathcal{A}$ one has

$$X^*_{\bar\lambda}(t)Q(t)X_\lambda(t) = G. \tag{1.1}$$

For any $\alpha, \beta \in \bar{\mathcal{I}}$, $\alpha \le \beta$, we use the notation $\Delta_\lambda(\alpha,\beta) = \int\limits_\alpha^\beta X^*_\lambda(t)w_\lambda(t)X_\lambda(t)dt$, $N = \{h \in \mathcal{H} | h \in Ker\Delta_\lambda(\alpha,\beta),\ \forall \alpha,\beta\}$, and let P be the orthogonal projection onto $N^\perp$.

Lemma 1.1. 1^o. *N is independent of $\lambda \in A$. Additionally, if $h \in N$, then $X_\lambda(t)h$ is a solution of*

$$\frac{i}{2}\left((Q(t)x(t))' + Q(t)x'(t)\right) = (ReH_i(t))x(t). \tag{1.2}$$

2^o. *Suppose that for a subset $F \subseteq \mathcal{H}$ the following condition holds:*

$$\exists \lambda_0 \in \mathcal{A}, \alpha, \beta \in \bar{\mathcal{I}},\ \textit{number}\ \delta > 0 : (\Delta_{\lambda_0}(\alpha,\beta)h,h) \ge \delta\|h\|^2, \quad \forall h \in F. \tag{1.3}$$

Let $c \in [\alpha,\beta]$. Then (1.3) is still valid if one replaces λ_0 with an arbitrary $\lambda \in \mathcal{A}$ and δ with some $\delta(\lambda) > 0$.

P r o o f o f 1^o. We are about to apply the following representation (to deduce it, use, for example [26, p. 36]):

$$H_\lambda(t) = A(t) + \lambda B(t) + \int\limits_{-\infty}^{\infty} \left\{\frac{1}{\tau-\lambda} - \frac{\tau}{1+\tau^2}\right\} d\sigma_t(\tau), \quad t \in \varepsilon, \lambda \in \mathcal{A}. \tag{1.4}$$

Here $A(t) = ReH_i(t)$, $0 \le B(t) \in B(\mathcal{H})$, for any fixed $t \in \varepsilon$ the operator-valued function $\sigma_t(\tau)$ is nondecreasing and $\int\limits_{-\infty}^{\infty} \frac{d(\sigma_t(\tau)g,g)}{1+\tau^2} < \infty$, $g \in \mathcal{H}$. Then

$$w_\lambda(t) = B(t) + \int\limits_{-\infty}^{\infty} \frac{d\sigma_t(\tau)}{|\tau-\lambda|^2}, t \in \varepsilon, \lambda \in \mathcal{A}. \tag{1.5}$$

Our subsequent argument requires

Proposition 1.1. *Suppose that for $g \in \mathcal{H}$ there exist $t_0 \in \varepsilon$ and $\lambda_0 \in \mathcal{A}$ such that $w_{\lambda_0}(t_0)g = 0$. Then for all $\lambda \in \mathcal{A}$ we have $w_\lambda(t_0)g = 0$, $h_\lambda(t_0)g = 0$ with $h_\lambda(t) = H_\lambda(t) - A(t)$.*

P r o o f o f P r o p o s i t i o n 1.1. Since the terms in (1.5) are nonnegative and $\sigma_{t_0}(\tau)$ is constant in neighborhoods of points from $\mathcal{A}\cap\mathbb{R}^1$, it follows from $w_{\lambda_0}(t_0)g=0$ that $B(t_0)g=0$, and there exists some $c>0$ such that $0=\int\limits_{-\infty}^{\infty}\frac{d(\sigma_{t_0}(\tau)g,g)}{|\tau-\lambda_0|^2}\geq c([\sigma_{t_0}(\tau_2)-\sigma_{t_0}(\tau_1)]g,g)$ with $-\infty<\tau_1<\tau_2<\infty$. Hence $d(\sigma_{t_0}(\tau)g,g)=0$, which, together with (1.4) and (1.5), implies our statement.

Turn back to the proof of 1^o. Let $h\in N$ at $\lambda=\lambda_0\in\mathcal{A}$. By a virtue of Prop. 1.1 for all $\lambda\in\mathcal{A}$ we have $h_\lambda(t)X_{\lambda_0}(t)h=0$ for a.a. $t\in\mathcal{I}$. This implies that for all $\lambda\in\mathcal{A}$, $X_{\lambda_0}(t)h$ is a solution of (1.2), and also a solution of (0.1) with $f(t)=0$. Hence there exists $g\in\mathcal{H}$ such that $X_{\lambda_0}(t)h=X_\lambda(t)g$ for $\lambda\in A$. Setting $t=c$, we deduce that $h=g$. Thus by Prop. 1.1, at a.a. $t\in\mathcal{I}$ one has $w_\lambda(t)X_\lambda(t)h=w_\lambda(t)X_{\lambda_0}(t)h=0$, that is, $h\in N$ for all $\lambda\in\mathcal{A}$, so 1^o is proved.

2^o. For $t\in\varepsilon$, $\lambda,\mu\in\mathcal{A}$, $g\in\mathcal{H}$, we have

$$\left\|\frac{H_\lambda(t)-H_\mu(t)}{\lambda-\mu}g\right\|$$

$$\leq\|B(t)g\|+\left(\int\limits_{-\infty}^{\infty}\frac{d(\sigma_t(\tau)g,g)}{|\tau-\lambda|^2}\right)^{1/2}\sup_{h\in\mathcal{H},\|h\|=1}\left(\int\limits_{-\infty}^{\infty}\frac{d(\sigma_t(\tau)h,h)}{|\tau-\mu|^2}\right)^{1/2},$$

whence, in view of (1.5), the following implication holds

$$(w_\lambda(t)g_n,g_n)\to 0\quad\Rightarrow\quad\frac{H_\lambda(t)-H_\mu(t)}{\lambda-\mu}g_n\to 0,\quad g_n\in\mathcal{H}.\tag{1.6}$$

Now assume there exist $\lambda\in\mathcal{A}$ and a sequence $\{h_n\}\in F$ with $\|h_n\|=1$ such that $(\Delta_\lambda(\alpha,\beta)h_n,h_n)\to 0$. Hence $\{h_n\}$ contains a subsequence $\{h_{n_j}\}$ such that at a.a. $t\in(\alpha,\beta)$ one has

$$(w_\lambda(t)X_\lambda(t)h_{n_j},X_\lambda(t)h_{n_j})\to 0.\tag{1.7}$$

Since the vector function $x_{n_j}(t)=X_\lambda(t)h_{n_j}$ is a solution of the equation

$$\frac{i}{2}\left[(Q(t)x(t))'+Q(t)x'(t)\right]-H_{\lambda_0}(t)x(t)=\left(H_\lambda(t)-H_{\lambda_0}(t)\right)x(t),$$

we have

$$x_{n_j}(t)=X_{\lambda_0}(t)\left[h_{n_j}+\int\limits_c^t X_{\lambda_0}^{-1}(s)(iQ(s))^{-1}\left[H_\lambda(s)-H_{\lambda_0}(s)\right]x_{n_j}(s)ds\right].\tag{1.8}$$

On the other hand, $x_{n_j}(t)\to 0$ in $L^2_{w_\lambda(t)}(\alpha,\beta)$, and the integral in (1.8) goes to zero uniformly in $t\in[\alpha,\beta]$ due to (1.7), (1.6) together with local B-integrability of $H_\lambda(t)$ since $c \in [\alpha,\beta]$. Therefore $X_{\lambda_0}h_{n_j} \to 0$ in $L^2_{w_\lambda}(\alpha,\beta)$ and hence in $L^2_{w_{\lambda_0}}(\alpha,\beta)$. This is because (1.5) implies that at any fixed $t \in \varepsilon$ the norms $(w_\lambda(t)f,f)^{1/2}$ are equivalent while λ is within any compact $K \subset \mathcal{A}$. Even more, the constants in inequalities between the norms do not depend on $t \in \varepsilon, \lambda \in K$. This contradicts (1.3). Lemma 1.1 is proved.

Lemma 1.1 with a constant leading coefficient in (0.1) can be found in [1].

For $x(t) \in \mathcal{H}$ or $x(t) \in B(\mathcal{H})$ introduce a notation $U[x(t)] = (Q(t)x(t), x(t))$ or $U[x(t)] = x^*(t)Q(t)x(t)$, respectively.

Definition 1.1. *An analytic operator-valued function $M(\lambda) = M^*(\bar\lambda) \in B(\mathcal{H})$ of nonreal λ is called a characteristic operator (c.o.) of the equation (0.1) on $\mathcal{I}$ (or briefly c.o.), if for $Im\lambda \neq 0$ and for any $\mathcal{H}$-valued vector function $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support the corresponding solution $x_\lambda(t)$ of (0.1) of the form*

$$x_\lambda(t) \equiv R_\lambda f = \int_{\mathcal{I}} X_\lambda(t)\left\{M(\lambda) - \frac{1}{2}sgn(s-t)(iG)^{-1}\right\}X^*_{\bar\lambda}(s)w_\lambda(s)f(s)ds \quad (1.9)$$

satisfies the condition

$$Im\lambda \lim_{(\alpha,\beta)\uparrow\mathcal{I}}(U[x_\lambda(\beta)] - U[x_\lambda(\alpha)]) \le 0,\ Im\lambda \neq 0. \quad (1.10)$$

Note that (1.10) is equivalent to claiming that for any finite $[\alpha,\beta] \supseteq \mathrm{supp} f(t)$ $((\alpha,\beta) \subseteq (a,b))$ one has

$$Im\lambda(U[x_\lambda(\beta)] - U[x_\lambda(\alpha)]) \le 0,\ Im\lambda \neq 0,^*$$

since the operator-valued function $Im\lambda X^*_\lambda(t)Q(t)X_\lambda(t)$ is nondecreasing for $Im\lambda \neq 0$. The latter is true because for any finite $(\alpha,\beta) \subseteq (a,b)$, $\lambda \in \mathcal{A}$, one has

$$X^*_\lambda(\beta)Q_\lambda(\beta)X_\lambda(\beta) - X^*_\lambda(\alpha)Q_\lambda(\alpha)X_\lambda(\alpha) = 2Im\lambda\Delta_\lambda(\alpha,\beta). \quad (1.11)$$

The following remark provides a relationship between c.o.'s and regular boundary problems for (0.1) with boundary conditions that depend on the spectral parameter.

Remark 1.1 (Cf. [1]). *Let $\mathcal{I} = (a,b)$ be a finite interval and suppose (1.3) is valid with $F = \mathcal{H}$.*

1^o. If the operator-valued functions $\mathcal{M}_\lambda, \mathcal{N}_\lambda$ depend analytically on nonreal λ,

$$\|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0, \quad (0 \neq h \in \mathcal{H}), \quad (1.12)$$

[*] One can readily use this remark to deduce that (1.10) holds for any $\mathcal{H}$-valued vector function $f(t) \in L^2_w(\mathcal{I})$ with compact support, as it holds for any piecewise-continuous vector function $f(t) \in \mathcal{H}$ with compact support.

and $\mathcal{L}_\lambda = \{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h | h \in \mathcal{H}\} \subset \mathcal{H}^2$ *are maximal* $Im\lambda diag(Q(a), -Q(b))$*-nonnegative subspaces in* $\mathcal{H}^2$, *(and so*

$$Im\lambda(\mathcal{N}_\lambda^* Q(b)\mathcal{N}_\lambda - \mathcal{M}_\lambda^* Q(a)\mathcal{M}_\lambda) \leq 0, \ Im\lambda \neq 0), \tag{1.13}$$

then the boundary problem given by attaching to (0.1) the boundary condition (in the form of [4])

$$\exists h = h(\lambda, f) \in \mathcal{H} : x(a) = \mathcal{M}_\lambda h \quad x(b) = \mathcal{N}_\lambda h \tag{1.14}$$

has for $Im\lambda \neq 0$ *a unique solution* $x_\lambda(t)$ *as in (1.9), with*

$$M(\lambda) = -\frac{1}{2}(X_\lambda^{-1}(a)\mathcal{M}_\lambda + X_\lambda^{-1}(b)\mathcal{N}_\lambda)(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda)^{-1}(iG)^{-1}, \tag{1.15}$$

and

$$(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda)^{-1} \in B(\mathcal{H}). \tag{1.16}$$

Here $M(\lambda) = M_\pm(\lambda)$ *for* $\pm Im\lambda > 0$, *with* $M_+(\lambda)$ *and* $M_-(\lambda)$ *being some (different in general) c.o.'s of (0.1) on* $\mathcal{I}$, *so that*

$$M_+(\lambda) = M_-(\lambda) \Leftrightarrow \mathcal{M}_{\bar\lambda}^* Q(a)\mathcal{M}_\lambda = \mathcal{N}_{\bar\lambda}^* Q(b)\mathcal{N}_\lambda, \quad Im\lambda \neq 0. \tag{1.17}$$

2 ^{o}. *If* $M(\lambda)$ *is a c.o. of (0.1) on* $\mathcal{I}$ *then* $x_\lambda(t)$ *is a solution of a boundary problem as in 1*^{o}.

P r o o f o f 1^o. Uniqueness of solution of the problem (0.1), (1.14), (1.13) follows from (1.11),(1.13), Lemma 1.1 and the condition (1.3) with $F = \mathcal{H}$.

Prove (1.16) assuming for certainty that $Im\lambda > 0$. Denote

$$T_\lambda = X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda.$$

Prove that $0 \notin \sigma_p(T_\lambda) \cup \sigma_c(T_\lambda)$. For if not, then for some $\lambda = \lambda_0 \in \mathbf{C}_+$ there exists a sequence of vectors $\{f_n\}$ such that $\|f_n\| = 1$, $T_{\lambda_0} f_n \to 0$, whence, with the notation $Y = X_{\lambda_0}(b)X_{\lambda_0}^{-1}(a)$, one has

$$\left(\left[\mathcal{N}_{\lambda_0}^* Q(b)\mathcal{N}_{\lambda_0} - \mathcal{M}_{\lambda_0}^* Q(a)\mathcal{M}_{\lambda_0}\right] f_n, f_n\right) + \left(\left[Q(a) - Y^* Q(b) Y\right] \mathcal{M}_{\lambda_0} f_n, \mathcal{M}_{\lambda_0} f_n\right)$$
$$= (Q(b)\mathcal{N}_{\lambda_0} f_n, \mathcal{N}_{\lambda_0} f_n) - (Q(b) Y \mathcal{M}_{\lambda_0} f_n, Y \mathcal{M}_{\lambda_0} f_n) \to 0. \tag{1.18}$$

Thus each term in the left hand side of (1.18) goes to 0 since both are nonpositive: the first one due to (1.13) and the second one due to (1.11). If so, $\mathcal{M}_{\lambda_0} f_n \to 0$ due to (1.11) and the condition (1.3) with $F = \mathcal{H}$, hence also $\mathcal{N}_{\lambda_0} f_n \to 0$. This implies that $Sf_n \to 0$, where the operator $S \in B(\mathcal{H})$ corresponds to $\mathcal{L}_{\lambda_0}$ in view of (2.4),(2.5). Since $\mathcal{L}_{\lambda_0}$ is a maximal $Im\lambda_0 diag(Q(a), -Q(b))$-nonnegative subspace, $R(S) = \mathcal{H}$ by Theorem 2.1. Furthermore, $Ker S = \{0\}$, as if for some nonzero $f \in \mathcal{H}$ $Sf = 0$, then by condition (2.6) of Th. 2.1 $S_1 f = 0 \Rightarrow \mathcal{M}_{\lambda_0} f =$

$\mathcal{N}_{\lambda_0} f = 0$, which contradicts (1.12). Thus $S^{-1} \in B(\mathcal{H})$ by the Banach theorem. The contradiction we get proves that $0 \notin \sigma_p(T_\lambda) \cup \sigma_c(T_\lambda)$.

Now prove that $0 \notin \sigma_r(T_\lambda)$. For if not, then there exist $\lambda_0 \in \mathbb{C}_+$ and a nonzero $f \in \mathcal{H}$ with $T^*_{\lambda_0} G f = 0$, whence by a virtue of (1.1)

$$\begin{aligned} 0 &= \mathcal{N}^*_{\lambda_0}(-Q(b))Q^{-1}(b)X^{*-1}_{\lambda_0}(b)Gf + \mathcal{M}^*_{\lambda_0}Q(a)Q^{-1}(a)X^{*-1}_{\lambda_0}(a)Gf \\ &= -\mathcal{N}^*_{\lambda_0}Q(b)X_{\bar{\lambda}_0}(b)f + \mathcal{M}^*_{\lambda_0}Q(a)X_{\bar{\lambda}_0}(a)f. \end{aligned}$$

Thus the vector $X_{\bar{\lambda}_0}(a)f \oplus X_{\bar{\lambda}_0}(b)f \in \mathcal{H}^2$ is $diag(Q(a), -Q(b))$-orthogonal to $\mathcal{L}_{\lambda_0}$, and hence is $diag(Q(a), -Q(b))$-nonpositive [25, p. 73]. On the other hand by (1.11) it is $diag(Q(a), -Q(b))$-nonnegative. Therefore $\Delta_{\bar{\lambda}_0}(a, b)f = 0$ by (1.11), whence $f = 0$ by Lemma 1.1 and the condition (1.3) with $F = \mathcal{H}$. That is $0 \notin \sigma_r(T_\lambda)$ and (1.16) is proved.

Now we are in a position to verify directly that (1.9), (1.15) is a solution of the problem (0.1), (1.14), (1.13); with $M_\pm(\lambda)$ being a c.o. by a virtue of (1.13) and 5^0 of Theorem 1.1 (our proof of this theorem does not elaborate Remark 1.1).

It follows from (1.1) that for $M(\lambda)$ (1.15) one has

$$M(\lambda) = M^*(\bar{\lambda}) \Leftrightarrow \mathcal{M}^*_{\bar{\lambda}} Q(a)\mathcal{M}_\lambda = \mathcal{N}^*_{\bar{\lambda}} Q(b)\mathcal{N}_\lambda, \tag{1.19}$$

hence statement (1.17), and 1^o is proved.

2^0. Let $M(\lambda)$ be a c.o. of (0.1). Represent $M(\lambda)$ in the form

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}. \tag{1.20}$$

Then $x_\lambda(t)$ (1.9),(1.20) is a solution of the problem (0.1), (1.14) with

$$\mathcal{M}_\lambda = X_\lambda(a)(\mathcal{P}(\lambda) - I), \qquad \mathcal{N}_\lambda = X_\lambda(b)\mathcal{P}(\lambda), \tag{1.21}$$

so that $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ (1.21) obviously satisfy (1.12); also by 3^0 of Theorems 1.1, 2.4 and Lemma 2.6, $\mathcal{L}_\lambda$ related to $\mathcal{M}_\lambda$, $\mathcal{N}_\lambda$ (1.21) is a maximal $Im\lambda diag(Q(a), -Q(b))$-nonnegative subspace. Remark 1.1 is proved completely.

Let $\Gamma(t)$ be an operator solution of $(Q(t)x(t))' + Q(t)x'(t) = 0$ normalized by the condition $\Gamma(c) = I$. We have

$$\Gamma(t), \Gamma^{-1}(t) \in B(\mathcal{H}), \Gamma(t) \in AC_{loc}, \Gamma^*(t)Q(t)\Gamma(t) = G. \tag{1.22}$$

Lemma 1.2. *The substitution $x(t) = \Gamma(t)y(t)$ reduces (0.1) to the equation*

$$iGy'(t) - \tilde{H}_\lambda(t)y(t) = \tilde{w}_\lambda(t)g(t), \quad t \in \bar{\mathcal{I}},^* \tag{1.23}$$

* A different substitution which reduces (0.1) to an equation with a constant leading coefficient is found in [27] (see also [3]).

with Nevanlinna's operator function

$$\tilde{H}_\lambda(t) = \Gamma^*(t)H_\lambda(t)\Gamma(t) \tag{1.24}$$

satisfying the same conditions as $H_\lambda(t)$*, and the weight*

$$\tilde{w}_\lambda(t) = \Gamma^*(t)w_\lambda(t)\Gamma(t), \tag{1.25}$$

$g(t) = \Gamma^{-1}(t)f(t)$.

*The equations (0.1) and (1.22)–(1.25) have the same c.o.'s.**

P r o o f. The relation (1.23) for $y(t) = \Gamma^{-1}(t)x(t)$ allows a direct verification.

Let $B(\mathcal{H}) \ni M(\lambda)$ be an arbitrary operator-valued function, $x_\lambda(t)$ (1.9) the associated solution of (0.1), $y_\lambda(t) = \Gamma^{-1}(t)x_\lambda(t)$. It is easy to see that $y_\lambda(t)$ is also given by (1.9) after substituting therein $X_\lambda(t)$ by the Cauchy operator for the equation (1.23) $Y_\lambda(t) = \Gamma^{-1}(t)X_\lambda(t)$, the weight $w_\lambda(t)$ by $\tilde{w}_\lambda(t)$ (1.25), and the vector function $f(t)$ by $g(t) = \Gamma^{-1}(t)f(t)$. Thus by a virtue of (1.22) $(Q(t)x_\lambda(t), x_\lambda(t)) = (Gy_\lambda(t), y_\lambda(t))$ for $t \in \bar{\mathcal{I}}$. It follows that if $M(\lambda)$ is a c.o. of (0.1), then $M(\lambda)$ is a c.o. of (1.22)–(1.25), and the converse is also true. The Lemma is proved.

Lemma 1.3. *Let* $A_j \in B(\mathcal{H})$, $j = 1, 2$. *Then: I.*

$$(A_1 \pm A_2)^{-1} \in B(\mathcal{H}), \tag{1.26}$$

if either of the following conditions holds:

1°.

$$(-1)^j A_j^* G A_j \le 0, \quad j = 1, 2, \tag{1.27}$$

the image $R(A_1)$ *or* $R(A_2)$ *is uniformly* G*-definite,*

$$\mathcal{L} = \{A_1 f \oplus A_2 f | f \in \mathcal{H}\} \subset \mathcal{H}^2 \tag{1.28}$$

is a maximal $G_2 = diag(G, -G)$*-nonnegative subspace in* $\mathcal{H}^2$*, and*

$$\|A_1 f\| + \|A_2 f\| > 0, \quad 0 \ne f \in \mathcal{H}. \tag{1.29}$$

2°. There exists a positive constant $\delta > 0$ *such that either*

a)

$$A_1^* G A_1 \ge 0, \quad (I - A_1^*)G(I - A_1) \le -\delta(I - A_1^*)(I - A_1), \tag{1.30}$$

$$A_2^* G A_2 \le -\delta A_2^* A_2, \quad (I - A_2^*)G(I - A_2) \ge 0,^{**} \tag{1.31}$$

*And, obviously, the same operators $\Delta_\lambda(\alpha, \beta)$.

**It is demonstrated in §2 that (1.30), (1.31) imply (even with $\delta = 0$) that A_1 and A_2 are projections.

or
b)

$$A_1^* G A_1 \geq 0, \quad (I - A_1^*) G (I - A_1) \leq 0, \tag{1.32}$$

$$A_2^* G A_2 \leq -\delta A_2^* A_2, \quad (I - A_2^*) G (I - A_2) \geq \delta (I - A_2^*)(I - A_2). \tag{1.33}$$

II. If $B_j \in \mathcal{H}$, $j = 1, 2$, then

$$(A_1 B_1 \pm A_2 B_2)^{-1} \in B(\mathcal{H}), \tag{1.34}$$

when

$$A_1^* G A_1 \geq 0, \quad (I - A_1^*) G (I - A_1) \leq 0, \tag{1.35}$$

and there exists such positive constant $\delta > 0$ that

$$A_2^* G A_2 \leq -\delta A_2^* A_2, \quad (I - A_2^*) G (I - A_2) \geq 0, \tag{1.36}$$

$$B_1^* G B_1 \geq \delta B_1^* B_1, \quad B_2^* G B_2 \leq -\delta B_2^* B_2, \tag{1.37}$$

$$\{B_1 f \oplus B_2 f | f \in \mathcal{H}\} \subset \mathcal{H}^2 \tag{1.38}$$

is a maximal G_2-nonnegative subspace in $\mathcal{H}^2$, and

$$\|B_1 f\| + \|B_2 f\| > 0, \quad 0 \neq f \in \mathcal{H}. \tag{1.39}$$

P r o o f. I.1^o. Prove (1.26) e.g., for $A_1 + A_2$. Suppose for certainty that $R(A_1)$ is uniformly G-definite, i.e.

$$\exists \delta > 0 : \quad A_1^* G A_1 \geq \delta A_1^* A_1. \tag{1.40}$$

Prove that

$$0 \notin \sigma_p(A_1 + A_2) \cup \sigma_c(A_1 + A_2). \tag{1.41}$$

If one assumes the contrary, then

$$\exists \{f_n\}, \quad f_n \in \mathcal{H}, \quad \|f_n\| = 1 : \ (A_1 + A_2) f_n \to 0, \tag{1.42}$$

whence

$$(A_1^* G A_1 f_n, f_n) + (-(A_2^* G A_2 f_n, f_n)) \to 0. \tag{1.43}$$

It follows that each term in the left hand side of (1.43) goes to 0, as both are nonnegative due to (1.27). Then by a virtue of (1.40), (1.42) one has

$$A_1 f_n \to 0, \quad A_2 f_n \to 0. \tag{1.44}$$

Therefore $S f_n \to 0$, with $S \in B(\mathcal{H})$ being associated to $\mathcal{L}$ (1.28) as in (2.4), (2.5). Since $\mathcal{L}$ (1.28) is maximal, $R(S) = \mathcal{H}$ by Th. 2.1. Besides that, $Ker S = \{0\}$, as if $Sf = 0$ for some nonzero $f \in \mathcal{H}$, then by the condition (2.6) of Th. 2.1 $S_1 f = 0$, hence $A_1 f = A_2 f = 0$, which contradicts (1.29). Therefore the Banach theorem implies $S^{-1} \in B(\mathcal{H})$. The contradiction we get proves (1.41).

Now prove that $0 \notin \sigma_r(A_1 + A_2)$. If not, then

$$\exists 0 \neq f \in \mathcal{H}: \quad (A_1^* + A_2^*)f = 0, \tag{1.45}$$

whence

$$\forall h \in \mathcal{H} \quad (GG^{-1}f, A_1 h) + (GG^{-1}f, A_2 h) = 0. \tag{1.46}$$

(1.46) means that

$$(G^{-1}f \oplus (-G^{-1}f)) \in \mathcal{L}^{[G_2]},$$

where $[A]$ stands for the A-orthogonal complement in the associated Hilbert space, whence by (1.27) and maximality of $\mathcal{L}$ (1.28) we deduce by Lemma 2.7 that

$$G^{-1}f \in \{A_1 h | h \in \mathcal{H}\}^{[G]}, \qquad G^{-1}f \in \{A_2 h | h \in \mathcal{H}\}^{[G]}. \tag{1.47}$$

In view of [25, p. 74] the first statement in (1.47), together with (1.40) implies that

$$\exists \delta_1 > 0: \quad (GG^{-1}f, G^{-1}f) \leq -\delta_1 \|G^{-1}f\|^2, \tag{1.48}$$

and the second statement in (1.47) implies that

$$(GG^{-1}f, G^{-1}f) \geq 0. \tag{1.49}$$

It follows from (1.48), (1.49) that $f = 0$, so that I.1^o is proved.

I.2^o. Assume for certainty that a) holds.

Prove (1.41). If (1.41) fails, then (1.42) holds, and hence (1.43) holds as well. Use the initial inequalities in (1.30), (1.31) to deduce from (1.43) that

$$(I - A_1)f_n - f_n \to 0, \quad (I - A_2)f_n - f_n \to 0, \tag{1.50}$$

in the same way as in the proof of (1.44), whence

$$((I - A_1^*)G(I - A_1)f_n, f_n)) + (-(I - A_2^*)G(I - A_2)f_n, f_n)) \to 0. \tag{1.51}$$

Now use the final inequalities in (1.30), (1.31) to deduce that $(I - A_1)f_n \to 0$ in the same way as in the proof of (1.44); this, together with (1.50) implies $f_n \to 0$, which is impossible. Thus (1.41) is proved.

Now prove $0 \notin \sigma_r(A_1 + A_2)$. If this fails, then (1.45) holds, whence

$$(G^{-1}A_1^* f, A_1^* f) = (G^{-1}A_2^* f, A_2^* f). \tag{1.52}$$

On the other hand, by Lemma 2.4 and Ths. 2.4, 2.7 one has $A_1^2 = A_1$ in view of (1.30). Therefore the vector $G^{-1}A_1^* f$ is G-orthogonal to $\{(I - A_1)h | h \in \mathcal{H}\}$, the latter subspace being in view of the final inequality in (1.30) maximal uniformly G-negative by theorem 2.4. Thus by [25, p. 74]

$$\exists \delta_1 > 0 : \quad (G^{-1}A_1^* f, A_1^* f) \geq \delta_1 \|G^{-1}A_1^* f\|^2.$$

In a similar way deduce that the right hand side of (1.52) is nonpositive, whence

$$A_1^* f = A_2^* f = 0. \tag{1.53}$$

But in this case (1.47) holds, and our proof can be completed in the same way as that of I.1^o, if one applies the initial inequalities in (1.30), (1.31).

II. Prove e.g., that $(A_1 B_1 + A_2 B_2)^{-1} \in B(\mathcal{H})$. Demonstrate first that

$$0 \notin \sigma_p(A_1 B_1 + A_2 B_2) \cup \sigma_c(A_1 B_1 + A_2 B_2). \tag{1.54}$$

If this fails, then

$$\exists \{f_n\}, \quad f_n \in \mathcal{H}, \quad \|f_n\| = 1 : \qquad (A_1 B_1 + A_2 B_2) f_n \to 0. \tag{1.55}$$

Use the initial inequalities in (1.35), (1.36), to deduce from (1.55) that

$$(I - A_1) B_1 f_n - B_1 f_n \to 0, \qquad (I - A_2) B_2 f_n - B_2 f_n \to 0, \tag{1.56}$$

in the same way as in the proof of (1.44), whence

$$-((I - A_1^*) G (I - A_1) B_1 f_n, B_1 f_n) + (B_1^* G B_1 f_n, f_n) \to 0, \tag{1.57}$$

$$((I - A_2^*) G (I - A_2) B_2 f_n, B_2 f_n) + (-(B_2^* G B_2 f_n, f_n)) \to 0. \tag{1.58}$$

Each term in (1.57) goes to zero, as they are nonnegative by the final inequality in (1.35) and the initial inequality in (1.37). Then in view of (1.37) one has

$$B_1 f_n \to 0. \tag{1.59}$$

In a similar way, it follows from (1.58), (1.36), (1.37) that

$$B_2 f_n \to 0. \tag{1.60}$$

In the same way as in the proof of I.1^o one can demonstrate that (1.59), (1.60) contradicts the maximality condition for the subspace (1.38) and the condition (1.39). This proves (1.54).

Now prove that $0 \notin \sigma_r(A_1 B_1 + A_2 B_2)$. If this fails, then

$$\exists 0 \neq f \in \mathcal{H} : \qquad (B_1^* A_1^* + B_2^* A_2^*) f = 0. \tag{1.61}$$

Use the maximality of (1.38), (1.37) deduce from (1.61) by Lemma 2.7 that

$$G^{-1}A_1^*f \in \{B_1h|h \in \mathcal{H}\}^{[G]}, \quad G^{-1}A_2^*f \in \{B_2h|h \in \mathcal{H}\}^{[G]},$$

in the same way as in the proof of I.1^o. From this we deduce again (1.53) by analogy with the proof of I.1^o and in view of (1.37). The proof finishes in the same way as that of I.1^o if one uses initial inequalities in (1.35), (1.36). The lemma is proved.

Note that in I.1^o of Lemma 1.3 one could not get rid of niether the condition of uniform G-definiteness for $R(A_1)$ or $R(A_2)$, nor the maximality condition for $\mathcal{L}$ (1.28).

In fact, (1.26) is not valid for

$$G = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

although all the assumptions of I.1^o hold except the uniform G-definiteness for $R(A_1)$ or $R(A_2)$.

Also for $\mathcal{H} = \mathbb{C} \oplus l^2$

$$G = diag(-1, 1, 1, \ldots), \qquad A_1 = diag(0, U), A_2 = (1, 0, 0, \ldots),$$

with U being the one-sided shift in l^2 (see [28]), (1.26) does not hold, although all assumptions of I.1^o but the maximality of $\mathcal{L}$ (1.28) are satisfied.

Note also that (1.26) fails if only (1.35), (1.36) are valid (and even $R(A_1)$ is uniformly G-definite). To see this, one should set

$$G = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}.$$

Thus introducing operators $B_j \neq I$ is necessary to make sure (1.34) is valid.

In view of [1] Lemma 1.3 provides an explicit expression for the projection onto $R(A_1)$ parallel to $R(A_2)$ in terms of A_1 and A_2.

Example 1.1. Suppose $\mathcal{I} = (a, b)$ is a finite interval, $a < c < b$, and the condition (1.3) with $F = \mathcal{H}$ holds for $\mathcal{I} = \mathcal{I}_- = (a, c)$ and for $\mathcal{I} = \mathcal{I}_+ = (c, b)$. Introduce the notation $Y_\lambda(t) = \Gamma^{-1}(t)X_\lambda(t)$, with $\Gamma(t)$ being as in (1.22). Then $Y_\lambda(a)$ and $Y_\lambda(b)$ are uniform $\pm G$-compressions and uniform $\pm G$-stretchings respectively as $\pm Im\lambda > 0$ due to (1.11), while $Y_\lambda^*(a)$ and $Y_\lambda^*(b)$ are uniform $\pm G^{-1}$-compressions and $\pm G^{-1}$-stretchings respectively due to (1.1), (1.11) as $\pm Im\lambda > 0$. Therefore [24, p. 66] with G indefinite, the operators $Y_\lambda(a)$ and $Y_\lambda(b)$ are unitarily dichotomic as $Im\lambda \neq 0$. Denote by $\mathcal{P}(Y_\lambda(a))$ and $\mathcal{P}(Y_\lambda(b))$ the Riesz projections for $Y_\lambda(a)$ and $Y_\lambda(b)$ corresponding to those parts of their spectra which are inside the unit circle. Let

$$\mathcal{H}_-(\lambda) = \mathcal{P}(Y_\lambda(a))\mathcal{H}, \qquad \mathcal{H}_+(\lambda) = \mathcal{P}(Y_\lambda(b))\mathcal{H}. \tag{1.62}$$

Since the subspaces $\mathcal{H}_-(\lambda), (I - \mathcal{P}(Y_\lambda(b)))\mathcal{H}$ and $\mathcal{H}_+(\lambda), (I - \mathcal{P}(Y_\lambda(a)))\mathcal{H}$ are respectively uniformly $\pm G$-positive and uniformly $\mp G$-positive as $\pm Im\lambda > 0$ by [24, p. 64], it follows from [25, p. 76] that

$$\mathcal{H} = \mathcal{H}_-(\lambda) \dot{+} \mathcal{H}_+(\lambda). \tag{1.63}$$

Denote by $\mathcal{P}(\lambda)$ the projection onto $\mathcal{H}_+(\lambda)$ parallel to $\mathcal{H}_-(\lambda)$. By I.2^o of Lemma 1.3, $(\mathcal{P}(Y_\lambda(b)) + \mathcal{P}(Y_\lambda(a)))^{-1} \in B(\mathcal{H})$, and hence by [1] (see also Sect. 2 of this work)

$$\mathcal{P}(\lambda) = \mathcal{P}(Y_\lambda(b))(\mathcal{P}(Y_\lambda(b)) + \mathcal{P}(Y_\lambda(a)))^{-1}, \tag{1.64}$$

which implies that $\mathcal{P}(\lambda)$ depends analytically on nonreal λ. Now it is easy to see in view (1.1), (1.19), (1.22) that the operator-valued function $M(\lambda)$ (1.20) with $\mathcal{P}$ as in (1.64), is a c.o. for (0.1) which corresponds (in terms of Remark 1.1) to boundary conditions like (1.14) with

$$\mathcal{M}_\lambda = -\Gamma(a)\mathcal{P}(Y_\lambda(a))Y_\lambda(a), \qquad \mathcal{N}_\lambda = \Gamma(b)\mathcal{P}(Y_\lambda(b))Y_\lambda(b).$$

Theorem 1.1^o. *Condition (1.10) for a the solution (1.9) as in the definition of c.o. is equivalent to the following inequality:*

$$\|R_\lambda f\|^2_{L^2_{w_\lambda}(\mathcal{I})} \le Im(R_\lambda f, f)_{L^2_{w_\lambda}(\mathcal{I})}/Im\lambda, \qquad Im\lambda \neq 0. \tag{1.65}$$

2^o. Condition (1.10) for an operator-valued function $M(\lambda) \in B(\mathcal{H})$ of the form (1.20) and arbitrary $\mathcal{H}$-valued vector functions $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support is equivalent to the following condition:

$$\forall [\alpha, \beta] \subseteq \bar{\mathcal{I}} : Im\lambda(\Gamma^+_\lambda(\beta) - \Gamma^-_\lambda(\alpha)) \le 0, \qquad Im\lambda \neq 0, \tag{1.66}$$

with

$$\Gamma^+_\lambda(t) = U[X_\lambda(t)\mathcal{P}G^{-1}P], \quad \Gamma^-_\lambda(t) = U[X_\lambda(t)(I - \mathcal{P})G^{-1}P]. \tag{1.67}$$

3^o. Condition (1.10) for an operator-valued function $M(\lambda) \in B(\mathcal{H})$ of the form (1.20) and arbitrary $\mathcal{H}$-valued vector functions $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support is equivalent to the following condition

$$\begin{aligned} PG^{-1}\left[(I - \mathcal{P}^*(\lambda))\Delta_\lambda(\alpha, c)(I - \mathcal{P}(\lambda)) + \mathcal{P}^*(\lambda)\Delta_\lambda(c, \beta)\mathcal{P}^*(\lambda)\right]G^{-1}P \\ \le \frac{Im\,[PM(\lambda)P]}{Im\lambda}, \qquad Im\lambda \neq 0 \end{aligned} \tag{1.68}$$

for any finite $\alpha \le c \le \beta$, $[\alpha, \beta] \subseteq \bar{\mathcal{I}}$. Hence for any c.o. $M(\lambda)$:

$$\frac{Im\,[PM(\lambda)P]}{Im\lambda} \ge 0, \quad Im\lambda \neq 0. \tag{1.69}$$

4°. If $M(\lambda)$ is a c.o. for (0.1), and an analytic operator-valued function of nonreal λ $M_0(\lambda) = M_0^*(\bar{\lambda}) \in B(\mathcal{H})$, $PM_0(\lambda)P = 0$, then $M(\lambda) + M_0(\lambda)$ is also a c.o., hence $PM(\lambda)P$ is a c.o as well.

5°. The validity of condition (1.10) for an operator-valued function $M(\lambda) \in B(\mathcal{H})$ and arbitrary $\mathcal{H}$-valued vector functions $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support in one of complex half-planes implies its validity for any such $f(t)$ in another half-planes, if one sets up $M(\bar{\lambda}) = M^*(\lambda)$ in (1.9).

6°. A c.o. in general is not uniquely determined by the problem (0.1), (1.10) (in particular for $P = I$); however, if $M_j(\lambda)$ are c.o.'s ($j = 1, 2$), and there exists a nonreal μ such that if $f \in \mathcal{H}$ and $\lim_{(\alpha,\beta)\uparrow\mathcal{I}}(\Delta_\mu(\alpha,\beta)f, f) < \infty$ implies $f \in N$, then $P[M_1(\mu) - M_2(\mu)]P = 0$. If one assumes additionally (1.3) with $F = \mathcal{H}$ then $M_1(\lambda) = M_2(\lambda)$, for all $\lambda \in C\backslash R^1$.

P r o o f o f 1°. To see that (1.10) and (1.65) are equivalent, note that for any solution $x(t)$ of (0.1) and any $[\alpha, \beta] \subseteq \bar{\mathcal{I}}$ one has for $Im\lambda \neq 0$

$$\|x(t)\|^2_{L^2_{w_\lambda}(\alpha,\beta)} - \frac{Im(x(t), f(t))_{L^2_{w_\lambda}(\alpha,\beta)}}{Im\lambda} = \frac{U[x(\beta)] - U[x(\alpha)]}{2Im\lambda}. \tag{1.70}$$

To prove 2° and other statements we need

Lemma 1.4. *Denote by $\mathcal{F}$ the set of $\mathcal{H}$-valued vector functions $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact supports $supp f(t) \subseteq \bar{\mathcal{I}}$, and define an operator*

$$I_\lambda f = \int_{\mathcal{I}} X^*_{\bar{\lambda}}(s) w_\lambda(s) f(s) ds, \tag{1.71}$$

which maps $\mathcal{F}$ into $\mathcal{H}$.

1) Let $[\alpha, \beta] \subseteq \bar{\mathcal{I}}$. Then

$$1) \quad \forall\lambda \in \mathcal{A}: \quad \Delta_{\bar{\lambda}}(\alpha,\beta)\mathcal{H} \subseteq I_\lambda\mathcal{F} \subseteq N^\perp, \tag{1.72}$$

and

$$\forall\lambda \in \mathcal{A}: \quad \overline{I_\lambda\mathcal{F}} = N^\perp. \tag{1.73}$$

2) Suppose there exist $\lambda_0 \in \mathcal{A}$ and $\alpha_0, \beta_0 \in \bar{\mathcal{I}}$ ($\alpha_0 \leq c \leq \beta_0$) such that $Ker\Delta_{\lambda_0}(\alpha_0, \beta_0) = N$. Then

$$\forall\lambda \in \mathcal{A}: \quad \overline{\Delta_\lambda(\alpha_0,\beta_0)\mathcal{H}} = N^\perp. \tag{1.74}$$

P r o o f o f L e m m a 1.4. 1) Setting for all $\lambda \in \mathcal{A}$, $h \in \mathcal{H}$

$$f(t) = \begin{cases} X_{\bar{\lambda}}(t)h, t \in [\alpha,\beta] \\ 0, t \notin [\alpha,\beta] \end{cases}, \tag{1.75}$$

we obtain $I_\lambda f = \Delta_{\bar\lambda}(\alpha,\beta)h$, and the left inclusion in (1.72) is proved. The right inclusion in (1.72) can be deduced by observing that if $h \in N$ then for all $\lambda \in \mathcal{A}$, $f(t) \in \mathcal{F}$ one has $(I_\lambda f, h) = \int\limits_{\mathcal{I}} (f(t), w_\lambda(t) X_{\bar\lambda}(t)h)dt = 0$, since $w_\lambda(t)X_{\bar\lambda}(t)h = 0$ (a.e.) in view of the definition of N and Lemma 1.1.

If (1.73) fails for some $\lambda = \lambda_0 \in \mathcal{A}$, then by (1.72) there exists a nonzero $g \in N^\perp$ such that for all $f(t) \in \mathcal{F}$ one has $(I_{\lambda_0} f, g) = 0$. Substituting $f(t)$ as in (1.75) with $\lambda = \lambda_0$, $h = g$, we obtain $\left(\Delta_{\bar\lambda_0}(\alpha,\beta)g, g\right) = 0$ for all $\alpha, \beta \in \bar{\mathcal{I}}$, whence $g \in N$ by Lemma 1.1. 1) is proved.

2) Follows from Lemma 1.1. Lemma 1.4 is proved.

Turn back to the proof of 2^o of Theorem 1.1. Fix a nonreal λ. Let $(\alpha_j, \beta_j) = \mathcal{I}_j \uparrow \mathcal{I}$, with the finite intervals $\mathcal{I}_j$ being such that $c \in \bar{\mathcal{I}}_j$. Introduce $N_j = Ker\Delta_\lambda(\alpha_j, \beta_j)$ and denote by Q_j the orthogonal projection onto N_j. As an non-increasing sequence of orthogonal projections, Q_j has a strong limit Q, which is again an orthogonal projection. Prove that Q projects onto N.

In fact, let $Qh = h$ for some $h \in \mathcal{H}$. Then $N_j \ni Q_j h \to Qh = h$. On the other hand, $N_j \subseteq N_k$ for $k \le j$, whence $Q_j h \in N_k$ for $k \le j$ and therefore $h \in N_k$ for all k, which implies $h \in \cap_k N_k = N$.

Let $M(\lambda)(1.20) \in B(\mathcal{H})$ satisfies (1.10) and $f \in N^\perp$. Introduce the notation $f_j = Q_j f$, $g_j = (I - Q_j)f$. One has $f = f_j \oplus g_j$, where $f_j \to 0$. On the other hand, by 2) of Lemma 1.4 for every g_j there exists a sequence $\{h_n^j\} \in \mathcal{H}$ such that $\Delta_{\bar\lambda}(\alpha_j, \beta_j)h_n^j \to g_j$. Hence in view of (1.11), when $\mathcal{I}_j \supseteq (\alpha, \beta)$:

$$\begin{aligned} Im\lambda \left\{U\left[X_\lambda(\beta)\mathcal{P}G^{-1}g_j\right] - U\left[X_\lambda(\alpha)(I-\mathcal{P})G^{-1}g_j\right]\right\} \\ \le \lim_{n\to\infty} Im\lambda\left\{U[x_\lambda(\beta_j)] - U[x_\lambda(\alpha_j)]\right\}, \end{aligned} \tag{1.76}$$

where $x_\lambda(t)$ is determined by (1.9) with $f(t)$ being as in (1.75) with $h = h_n^j$, $[\alpha,\beta] = [\alpha_j, \beta_j]$. In view of (1.10), the left hand side of (1.76) is nonpositive for $Im\lambda \ne 0$, whence by (1.76) $Im\lambda\left\{U\left[X_\lambda(\beta)\mathcal{P}G^{-1}f\right] - U\left[X_\lambda(\alpha)(I-\mathcal{P})G^{-1}f\right]\right\} \le 0$ for all $f \in N^\perp$. Now (1.66) is proved for $M(\lambda)$.

Conversely, suppose that an operator $M(\lambda) \in B(\mathcal{H})$ as in (1.20) satisfy (1.66). Then the corresponding solutions $x_\lambda(t)$ (1.9) satisfy (1.10) by 1) of Lemma 1.4, which proves 2^o.

3^o follows from the fact that for all $M(\lambda)(1.20) \in B(\mathcal{H})$ one has

$$\begin{aligned} (\Gamma^+_\lambda(\beta) - \Gamma^-_\lambda(\alpha)) + 2Im[PM(\lambda)P] \\ = 2Im\lambda PG^{-1}[(I-\mathcal{P}^*(\lambda))\Delta_\lambda(\alpha,c)(I-\mathcal{P})) + \mathcal{P}^*(\lambda)\Delta_\lambda(c,\beta)\mathcal{P}(\lambda)]G^{-1}P, \end{aligned} \tag{1.77}$$

together with 2^0.

4^o. Take into account that by (1.11) $X^*_\lambda(\alpha)Q(\alpha)X_\lambda(\alpha)h_0 = X^*_\lambda(\beta)Q(\beta)$

$\times X_\lambda(\beta)h_0$, for all $h_0 \in N$ it is easy to verify that

$$U[X_\lambda(\beta)(\mathcal{P}(\lambda)G^{-1}Pf + (I-P)g)] - U[X_\lambda(\alpha)((\mathcal{P}(\lambda)-I)G^{-1}Pf + (I-P)g)] \\ = (\Gamma^+_\lambda(\beta)f, f) - (\Gamma^-_\lambda(\alpha)f, f), \quad \forall f, g \in \mathcal{H}. \tag{1.78}$$

Now represent $M(\lambda)$ in the form (1.20), and $M(\lambda) + M_0(\lambda)$ in the form $M(\lambda) + M_0(\lambda) = \left(\mathcal{P}(\lambda) + \Delta\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}$, then $M_0 = \Delta\mathcal{P}(\lambda)(iG)^{-1}$ hence $P\Delta\mathcal{P}(\lambda)(iG)^{-1}P = 0$, so 4^o follows from (1.78) and 2^0.

5^o. Suppose for certainty that for $Im\lambda > 0$, $M(\lambda) \in B(\mathcal{H})$ satisfies (1.10), and $\alpha, \beta \in \bar{\mathcal{I}}$, $\alpha \le c \le \beta$. Denote by P_1 an analogue for the orthogonal projection P with respect to $\mathcal{I}_1 = (\alpha, \beta)$ and represent $M_1(\lambda) = P_1 M(\lambda) P_1$ in the form

$$M_1(\lambda) = \left(\mathcal{P}_1(\lambda) - \frac{1}{2}I\right)(iG)^{-1}. \tag{1.79}$$

Then by (1.11), 3^0 of Th. 1.1, Th. 2.4 and Lem. 2.6

$$\mathcal{L}_\lambda = \{X_\lambda(\alpha)\left[(\mathcal{P}_1(\lambda) - I)(iG)^{-1}P_1 + I - P_1\right] f \oplus X_\lambda(\beta) \\ \times \left[\mathcal{P}_1(\lambda)(iG)^{-1}P_1 + I - P_1\right] f | f \in \mathcal{H}\} \tag{1.80}$$

is a maximal $diag(Q(\alpha), -Q(\beta))$-nonnegative subspace. Therefore by Theorem 2.5, $\mathcal{L}_{\bar{\lambda}}$ determined by (1.80) after substituting $\mathcal{P}_1(\lambda)$ with $\mathcal{P}_1(\bar{\lambda}) =^{def} G^{-1}(I - \mathcal{P}_1^*)(\lambda))G$ is $diag(Q(\alpha), -Q(\beta))$-nonpositive.

Therefore by Lemma 1.1, (1.78), 2^0 of Theorem 1.1, (1.10) is valid for $Im\lambda < 0$ if one replaces in (1.10) $\mathcal{I} = \mathcal{I}_1$, and also replaces in (1.9) $M(\lambda)$ with $M_1^*(\lambda) = P_1 M^*(\lambda) P_1$, hence even with $M^*(\lambda)$ by 4^0 of Theorem 1.1. Thus 5^o is proved for any such $\mathcal{H}$-valued vector function $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support that $suppf(t) \subseteq [\alpha, \beta]$, and hence due to arbitrariness of α, β for any $\mathcal{H}$-valued $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ with compact support as well.

6^0. Let $M_1(\lambda)$ and $M_2(\lambda)$ be c.o.'s. Denote by $R^1_\lambda f$ and $R^2_\lambda f$ the corresponding solutions of (0.1) of the form (1.9). By a virtue of (1.65) for c.o. that has already been proved, one has $R^1_{\mu_0} f - R^2_{\mu_0} f = X_{\mu_0}(t)\left[M_1(\mu_0) - M_2(\mu_0)\right] I_{\mu_0} f \in L^2_{w_{\mu_0}}(\mathcal{I})$, with $I_\lambda f$ (see (1.71)). So the initial statement of 6^o follows from (1.73).

Now the final statement follows from $M(\bar{\lambda}) = M^*(\lambda)$, together with the following lemma, which could also make an independent interest.

Lemma 1.5. *Suppose (1.3) holds with $F = \mathcal{H}$, and for some $\mu \in C \backslash R^1$ one has*

$$h \in \mathcal{H}, \lim_{(\alpha,\beta)\uparrow\mathcal{I}} (\Delta_\mu(\alpha, \beta)h, h) < \infty \Rightarrow h = 0. \tag{1.81}$$

Then (1.81) also holds with μ being replaced by any such λ that $Im\lambda Im\mu > 0$.

P r o o f o f L e m m a 1.5. Suppose there exists $\mu_0 \in C\backslash R^1$ ($\mu_0 \neq \mu$, $Im\mu_0 Im\mu > 0$) such that the equation $l[y] = H_{\mu_0}(t)y$ has a solution $0 \not\equiv y(t) \in L^2_{w_{\mu_0}}(\mathcal{I})$ (and hence $y(t) \in L^2_{w_\lambda}(\mathcal{I}), \lambda \in \mathcal{A}$), with $l[y] = \frac{i}{2}((Q(t)y)' + Q(t)y')$. Consider the nonhomogeneous equation

$$l[z] - H_\mu(t)z = (H_{\mu_0}(t) - H_\mu(t))y. \tag{1.82}$$

Every its solution can be represented in the form

$$z(t) = y(t) + X_\mu(t)g, g \in \mathcal{H}, \tag{1.83}$$

so that $z(t) \notin L^2_{w_\mu}(\mathcal{I})$ for $g \neq 0$.

Let $M(\lambda)$ be a c.o. for (0.1) (a proof of its existence does not elaborate of Th. 1.1).

Use property (1.65) of c.o., (1.3) with $F = \mathcal{H}$, and the functions of the form (1.75) to demonstrate that there exist constants $c_j(\lambda, \tau)$ such that

$$\forall \lambda \in \mathbb{C}\backslash R^1, h \in \mathcal{H} : \|X_\lambda(t)(\mathcal{P}(\lambda) - I)h\|_{L^2_{w_\lambda}(a,\tau)} \leq c_1(\lambda, \tau)\|h\|,$$

$$\|X_\lambda(t)\mathcal{P}(\lambda)h\|_{L^2_{w_\lambda}(\tau,b)} \leq c_2(\lambda, \tau)\|h\|$$

with $\tau \in \bar{\mathcal{I}}$, $\mathcal{P}(\lambda)$ (see (1.20)).

It follows that there exists a constant $k(s, \lambda)$ such that

$$\int_{\mathcal{I}} (w_\lambda(t)K(t, s, \lambda)h, K(t, s, \lambda)h)dt \leq k(s, \lambda)\|h\|^2 \tag{1.84}$$

with

$$K(t, s, \lambda) = X_\lambda(t)\left\{M(\lambda) - \frac{1}{2}sgn(s - t)(iG)^{-1}\right\}X^*_{\bar{\lambda}}(s). \tag{1.85}$$

It follows from (1.84) and representation (1.5) for the weight $w_\lambda(t)$ (cf. the proof of 2^0, Prop. 1.1), that the integral

$$x(t) = \int_{\mathcal{I}} K(t, s, \mu)f(s)ds \tag{1.86}$$

converges strongly, with $f(t) = (H_{\mu_0}(t) - H_\mu(t))v(t)$, where measurable $v(t) \in L^2_{w_\mu}(\mathcal{I})$.

Clearly, $x(t)$ (1.86) is a solution of (1.82) with $y(t)$ being replaced by $v(t)$. In the case when $v(t)$ has a compact support $suppf \subseteq [\alpha, \beta] \subseteq \bar{\mathcal{I}}$, an argument similar to that of the proof of 1^o demonstrates

$$\|x(t)\|^2_{L_{w_\mu}(\alpha,\beta)} \leq \frac{Im \int_\alpha^\beta (x(t), f(t))dt}{Im\mu}$$

since $M(\lambda)$ is a c.o.

One deduces that in general case $x(t) \in L^2_{w_\mu}(\mathcal{I})$ (hence in view of (1.83) $x(t) = y(t)$ when $v(t) = y(t)$). Now it is easy to demonstrate that there exists a constant c such that

$$\|x(t) - x_n(t)\|_{L^2_{w_\mu(\mathcal{I})}} \leq c\|v(t) - v_n(t)\|_{L^2_{w_\mu(\mathcal{I})}},$$

where $v_n(t) = \chi_n(t)v(t)$, $\chi_n(t)$ are characteristic functions of finite intervals $(\alpha_n, \beta_n) \uparrow \mathcal{I}$, $x_n(t)$ the associated solutions of the form (1.86). On the other hand

$$\frac{1}{2}\left|(Q(t)x(t), x(t))\,|_\alpha^\beta - (Q(t)x_n(t), x_n(t))\,|_\alpha^\beta\right|$$
$$\leq |Im\mu|\left|\|x(t)\|^2_{L^2_{w_\mu}(\alpha,\beta)} - \|x_n(t)\|^2_{L^2_{w_\mu}(\alpha,\beta)}\right|$$
$$+\left|Im\int_\alpha^\beta [(x(t), f(t)) - (x_n(t), f_n(t))]\,dt\right| \to 0$$

uniformly in α, $\beta \in \bar{\mathcal{I}}$. Furthermore,

$$Im\mu\,(Q(t)x_n(t), x_n(t))\,|_{\alpha_n}^{\beta_n} \leq 0$$

since $M(\lambda)$ is a c.o. Hence

$$\lim_{(\alpha_n,\beta_n)\uparrow\mathcal{I}} Im\mu\,(Q(t)x(t), x(t))\,|_{\alpha_n}^{\beta_n} \leq 0$$

(the limit exists since $x(t), v(t) \in L^2_{w_\mu}(\mathcal{I})$). In particular,

$$\lim_{(\alpha_n,\beta_n)\uparrow\mathcal{I}} Im\mu\,(Q(t)y(t), y(t))\,|_{\alpha_n}^{\beta_n} \leq 0. \tag{1.87}$$

On the other hand by (1.11), one has

$$\|y(t)\|^2_{L^2_{w_{\mu_0}}(\alpha,\beta)} = \frac{(Q(t)y(t), y(t))|_\alpha^\beta}{2Im\mu_0},$$

whence $\|y(t)\|_{L^2_{w_\mu}(\mathcal{I})} = 0$ in view of (1.87) and $Im\mu_0 Im\mu > 0$. Thus $y(t) \equiv 0$, which is due to (1.3) with $F = \mathcal{H}$. Now Lemma 1.5, together with 6^o of Th. 1.1, are proved.

Conditions of existence for c.o. are given by

Theorem 1.2. *A c.o. of (0.1) on $\mathcal{I}$ exists, if either one of the ends of $\mathcal{I}$ is finite or if for some $\lambda_0 \in A \cap R^1$ the norm $\|X^*_{\lambda_0}(t)w_{\lambda_0}(t)X_{\lambda_0(t)}\|$ is summable at one of the ends of $\mathcal{I}$. Also, a c.o. exists if (1.3) holds with $F = N^\perp$.*

To prove Theorem 1.2, we need the following lemma, which could possibly make an independent interest.

Lemma 1.6. *Suppose that operator-valued functions $M_n(\lambda)$ are c.o. for (0.1) on finite or infinite intervals $\mathcal{I}_n$ such that $c \in \bar{\mathcal{I}}_n$, $\mathcal{I}_n \uparrow \mathcal{I}$. Assume also that for any compact $K \subset C \backslash R^1$*

$$\exists c(K) : \forall \lambda \in K \qquad \|M_n(\lambda)\| < c(K). \tag{1.88}$$

Then there is a subsequence $\{n_j\}$ such that for $Im\lambda \neq 0$ the limit

$$w - \lim M_{n_j}(\lambda) = M(\lambda) \tag{1.89}$$

exists and $M(\lambda)$ is a c.o. for (0.1) on $\mathcal{I}$.

P r o o f o f L e m m a 1.6. One can deduce from Vitali's theorem [29] and weak compactness of the family $M_n(\lambda)$ at any fixed nonreal λ (which is itself due to (1.88)) that there exists a subsequence $\{n_j\}$ such that (1.89) holds, with the operator-valued function $M(\lambda) = M^*(\bar{\lambda})$ being analytic for nonreal λ.

Prove that $M(\lambda)$ is a c.o. for (0.1) on $\mathcal{I}$. By (1.77) (which is valid for all $M(\lambda)$(1.20)$\in B(\mathcal{H})$) one has

$$\forall [\alpha,\beta] \subset \bar{\mathcal{I}}, \alpha \le c \le \beta, f \in \mathcal{H} \quad \frac{(\Gamma_\lambda^+(\beta)f,f) - (\Gamma_\lambda^-(\alpha)f,f)}{Im\lambda}$$
$$= 2[(\Delta_\lambda(\alpha,c)(I-\mathcal{P}(\lambda))G^{-1}Pf,(I-\mathcal{P}(\lambda))G^{-1}Pf)$$
$$+(\Delta_\lambda(c,\beta)\mathcal{P}(\lambda)G^{-1}Pf,\mathcal{P}(\lambda)G^{-1}Pf) - \frac{Im(PMPf,f)}{Im\lambda}]. \tag{1.90}$$

Denote by P_n an analog with respect to $\mathcal{I}_n$ for the orthogonal projection P and by $\mathcal{P}_n(\lambda)$ an analog $\mathcal{P}(\lambda)$ (1.20) for $M_n(\lambda)$. Use nonnegativity of $\Delta_\lambda(s,t)$, [30, pp. 176, 193], and the fact that $P_n \to^s P$ to deduce from (1.90):

$$\forall [\alpha,\beta] \subset \bar{\mathcal{I}},\ \alpha \le c \le \beta, f \in \mathcal{H}: \quad \frac{(\Gamma_\lambda^+(\beta)f,f) - (\Gamma_\lambda^-(\alpha)f,f)}{Im\lambda}$$
$$\le 2\big[\underline{\lim}(\Delta_\lambda(\alpha,c)(I-\mathcal{P}_n(\lambda))G^{-1}P_nf,(I-\mathcal{P}_n(\lambda))G^{-1}P_nf)$$
$$+\underline{\lim}(\Delta_\lambda(c,\beta)\mathcal{P}_n(\lambda)G^{-1}P_nf,\mathcal{P}_n(\lambda)G^{-1}P_nf) - \lim \frac{Im(P_nM_n(\lambda)P_nf,f)}{Im\lambda}\big]$$
$$\le 2\underline{\lim}\big[(\Delta_\lambda(\alpha,c)(I-\mathcal{P}_n(\lambda))G^{-1}P_nf,(I-\mathcal{P}_n(\lambda))G^{-1}P_nf)$$
$$+(\Delta_\lambda(c,\beta)\mathcal{P}_n(\lambda)G^{-1}P_nf,\mathcal{P}_n(\lambda)G^{-1}P_nf) - \frac{Im(P_nM_n(\lambda)P_nf,f)}{Im\lambda}\big] \le 0 \tag{1.91}$$

since by 3^0 of Th. 1.1 the expression in brackets is nonpositive because $M_n(\lambda)$ is a c.o. on the interval $\mathcal{I}_n$, which contains (α, β) for n large enough. Thus for $M(\lambda)$ (1.89), (1.66) holds if $[\alpha, \beta] \subset \bar{\mathcal{I}}$ and, hence, if $[\alpha, \beta] \subseteq \bar{\mathcal{I}}$ (in view, that $\Gamma_\lambda^{\pm}(t)$ depends on t continuously). Now by 2^o of Th. 1.1, Lemma 1.6 is proved.

P r o o f o f T h e o r e m 1.2. Let G is definite. Then by 2^o of Theorem 1.1 $M(\lambda)$ (1.20) with $\mathcal{P} = \frac{1}{2}(I - sgn(Im\lambda G))$ is a c.o. of (0.1). Let G is indefinite.

By Lemma 1.2, it suffices to prove the theorem for (0.1) with constant

$$Q(t) = G. \tag{1.92}$$

Case I. Suppose one of the ends of $\mathcal{I}$ is finite, e.g. $a > -\infty$. Consider (0.1), (1.92) with the operator-valued coefficient $H_\lambda(t)$ being replaced by

$$H_\lambda^n(t) = \begin{cases} H_\lambda, a < t < \beta_n = \min\{b, n\} \\ \lambda I, \beta_n < t < \infty, \end{cases} \tag{1.93}$$

with $n \geq c$. Then the Cauchy operator $X_\lambda(t)$ is to be replaced by

$$X_\lambda^n(t) = \begin{cases} X_\lambda(t), & a < t < \beta_n \\ e^{(iG)^{-1}\lambda(t-\beta_n)} X_\lambda(\beta_n), & \beta_n \leq t < \infty \end{cases}. \tag{1.94}$$

Using an argument similar to that use with $Y_\lambda(b)$ in Ex. 1.1, we observe that for $Im\lambda \neq 0$, $t > \beta_n$, the operators $Y_\lambda^n(t) = X_\lambda^n(t) X_\lambda^{-1}(a)$ are unitarily dichotomic. Consider the subspaces

$$\mathcal{H}_+^n(\lambda) = \mathcal{P}(Y_\lambda^n(\beta_n + 1))\mathcal{H}$$

similar to $\mathcal{H}_+(\lambda)$ in Ex. 1.1. Consider one more projection-valued function

$$\Pi(\lambda) = \begin{cases} \Pi, & Im\lambda > 0, \\ I - \Pi, & Im\lambda < 0, \end{cases} \tag{1.95}$$

where Π project onto fixed maximal uniformly G-positive subspace parallel to its G-orthogonal complement. The latter subspace is maximal uniformly G-negative by [25, p. 74].

Therefore in view of [25, p. 76] one has

$$\mathcal{H} = \Pi(\lambda)\mathcal{H} \dot{+} \mathcal{H}_+^n(\lambda).$$

Denote by $\mathcal{P}_n(\lambda)$ the projection onto $X_\lambda^{-1}(a)\mathcal{H}_+^n(\lambda)$ parallel to $X_\lambda^{-1}(a)\Pi(\lambda)\mathcal{H}$. By Lemma 1.3

$$\left(\mathcal{P}\left(Y_\lambda^n(\beta_n + 1)\right) + \Pi(\lambda)\right)^{-1} \in B(\mathcal{H}).$$

Therefore by [1] (see also Sect. 2)

$$\mathcal{P}_n(\lambda) = X_\lambda^{-1}(a)\mathcal{P}\left(Y_\lambda^n(\beta_n+1)\right)\left(\mathcal{P}\left(Y_\lambda^n(\beta_n+1)\right)+\Pi(\lambda)\right)^{-1}X_\lambda(a) \qquad (1.96)$$

depends analytically on nonreal λ. It is easy to see that in view of (1.95), (1.1), (1.19)

$$M_n(\lambda) = \left(\mathcal{P}_n(\lambda) - \frac{1}{2}\right)(iG)^{-1} \qquad (1.97)$$

is a c.o. for (0.1), (1.92) with $H_\lambda(t) = H_\lambda^n(t)$ (1.93) on $(a, \beta_n + 1)$, hence also for (0.1), (1.92) on $\mathcal{I} = \mathcal{I}_n = (a, \beta_n)$. Since n is arbitrary, the theorem is proved for a finite interval (a, b).

Proceed with proving the theorem for the case $b = \infty$. Due to the uniform $\pm G$-positivity of $\Pi(\lambda)\mathcal{H}$ and the uniform $\pm G$-negativity of $\mathcal{H}_+^n(\lambda)$ for $Im\lambda \neq 0$, the mutual inclination (see [24]) of these subspaces at any fixed nonreal λ is separated out from zero uniformly in n. Therefore [24, p. 224] for any compact $K \subset \mathbb{C}\backslash R^1$ one has

$$\exists C(K) : \forall\lambda \in K \quad \|\mathcal{P}_n(\lambda)\| \le C(K). \qquad (1.98)$$

Now the statement of the theorem in the Case I follows from (1.98), (1.97), and Lemma 1.6.

Case II. Assume, for example, $a = -\infty$,

$$\exists\lambda_0 \in \mathcal{A} \cap R^1 : \quad \int_{-\infty}^{c} \left\|X_{\lambda_0}^*(t)w_{\lambda_0}(t)X_{\lambda_0}(t)\right\| dt < \infty. \qquad (1.99)$$

Lemma 1.7. *If (1.99) holds, then the substitution $x(t) = X_{\lambda_0}(t)z(t)$ reduces equation (0.1) with $f(t) = 0$ to the equation*

$$iGz'(t) - X_{\lambda_0}^*(t)\left(H_\lambda(t) - H_{\lambda_0}(t)\right)X_{\lambda_0}(t)z(t) = 0,\ t \in \bar{\mathcal{I}}. \qquad (1.100)$$

The equation (0.1) and its analog for (1.100) have the same c.o.'s. Also, for the Cauchy operator $Z_\lambda(t) = X_{\lambda_0}^{-1}(t)X_\lambda(t)$ of the equation (1.100)

$$\exists u - \lim_{t\to-\infty} Z_\lambda(t) = Z_\lambda(\infty),$$

where $Z_\lambda(\infty)$ depends analytically on $\lambda \in \mathcal{A}$ and $Z_\lambda^{-1}(\infty) \in B(\mathcal{H})$.

P r o o f o f L e m m a 1.7. The proof of coincidence of c.o. for (0.1) and (1.100) is the same as that of Lemma 1.2.

An estimate for $\left\|X_{\lambda_0}^*(t)\left(H_\lambda(t) - H_{\lambda_0}(t)\right)X_{\lambda_0}(t)\right\|$, like that preceding (1.6) demonstrates in view of (1.5), (1.99):

$$\int_{-\infty}^{c} \left\|X_{\lambda_0}^*(t)\left(H_\lambda(t) - H_{\lambda_0}(t)\right)X_{\lambda_0}(t)\right\| dt < \infty,$$

whence by [24, p. 166], [31] the statement of the lemma with respect to $Z_\lambda(t)$ follows. Lemma 1.7 is proved.

Now in view of Lemma 1.7, c.o. for (1.100) (hence also c.o. for (0.1)) can be produced similarly to the Case I with $X_\lambda(a)$ being replaced by $Z_\lambda(\infty)$.

Case III. Suppose (1.3) holds for $F = N^\perp$.

Lemma 1.8. *Suppose that (1.3) holds for $F = N^\perp$ and in (1.3) either $(\alpha, \beta) \subseteq (a, c)$ or $(\alpha, \beta) \subseteq (c, b)$. Let $M'(\lambda)$ be a c.o. for (0.1) on the interval $\mathcal{I}' \subseteq \mathcal{I}$ that contains, respectively, either (α, c) or (c, β). Then for any compact $K \subset \mathbb{C} \backslash R^1$ there exists a constant $c = c(K)$ independent of $M'(\lambda)$ and $\mathcal{I}'$ and such that*

$$\forall \lambda \in K: \quad \left\| PM'(\lambda)P \right\| \le c(K).$$

P r o o f of the lemma follows from 3^o of Th. 1.1.

In view of the Cases I and II of the theorem that have already been proved, it suffices to prove Case III for $\mathcal{I} = (-\infty, \infty)$.

First assume $c \notin (\alpha, \beta)$, e.g. for certainty $c \le \alpha$. Hence (1.3) remains valid if one replaces therein (α, β) with (c, β). Similarly to (1.93) and (1.94), extend $H_\lambda(t)$ and $X_\lambda(t)$ from $(-n, n) \supseteq (c, \beta)$ onto $(-n-1, n+1)$ and use the argument of Ex. 1.1 and Case I to obtain

$$\mathcal{H} = \mathcal{P}\left(X_\lambda^n(-n-1)\right)\mathcal{H} \dotplus \mathcal{P}\left(X_\lambda^n(n+1)\right)\mathcal{H}.$$

Denote by $\mathcal{P}_n(\lambda)$ the projection onto $\mathcal{P}\left(X_\lambda^n(n+1)\right)\mathcal{H}$ parallel to $\mathcal{P}\left(X_\lambda^n(-n-1)\right)\mathcal{H}$. Similarly to Ex. 1.1 and Case I

$$\mathcal{P}_n(\lambda) = \mathcal{P}\left(X_\lambda^n(n+1)\right)\left(\mathcal{P}\left(X_\lambda^n(n+1)\right) + \mathcal{P}\left(X_\lambda^n(-n-1)\right)\right)^{-1}, \tag{1.101}$$

with $(...)^{-1} \in B(\mathcal{H})$.

It is easy to see that, similarly to Case I, M_n (1.20), (1.101) is a c.o. for (0.1), (1.92) on $(-n, n)$. By 4^o of Th. 1.1 $PM_n(\lambda)P$ is also a c.o. for (0.1) on $(-n, n)$. Now the proof for the case $c \notin (\alpha, \beta)$ follows from Lemmas 1.8, 1.6.

Finally, let $c \in (\alpha, \beta)$. It is easy to see that by (1.3) with $F = \mathcal{H}^\perp$ one has for $\lambda \in \mathcal{A}$:

$$\exists \delta_1 = \delta_1(\lambda) > 0: \quad (X_\lambda^{*-1}(\alpha)\Delta_\lambda(\alpha, \beta)X_\lambda^{-1}(\alpha)f, f) \ge \delta_1 \|f\|^2, \quad \forall f \in [X_\lambda(\alpha)N]^\perp$$

($X_\lambda(\alpha)N$ does not depend on $\lambda \in \mathcal{A}$ by Lem. 1.1).

Thus it follows from the above observations that there exists a c.o. $\tilde{M}(\lambda)$ for (0.1) if its Cauchy operator is normalized as the identity not at c but at α (i.e. $X_\lambda(t)$ is replaced by $X_\lambda(t)X_\lambda^{-1}(\alpha)$). But at that case, as one can easily see, $M(\lambda) = X_\lambda^{-1}(\alpha)\tilde{M}(\lambda)X_{\bar{\lambda}}^{*-1}(\alpha)$ is a c.o. for (0.1). Theorem 1.2 is proved.

Note that in Th. 1.2 in the case $\mathcal{I} = R^1$ it is impossible in general to get rid of (1.3) with $F = N^{\perp}$, as one can see from

Example 1.2. Consider the block-diagonal equation (0.1) with

$$Q(t) = diag\{G_k\}_{k=1}^{\infty}, \quad H_\lambda(t) = diag\{H_\lambda^k(t)\}_{k=1}^{\infty}, \tag{1.102}$$

with

$$G_k = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad H_\lambda^k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} \mu_k^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 < \mu_k \to 0. \tag{1.103}$$

(1.10) is valid for any solution of the form (1.9) for equations (0.1), (1.102), (1.103) on $\mathcal{I} = R^1$ if and only if in (1.9)

$$M(\lambda) = \frac{i}{2} diag\left\{\begin{pmatrix} \frac{1}{\mu_k\sqrt{\lambda}} & 0 \\ 0 & \mu_k\sqrt{\lambda} \end{pmatrix}\right\}_{k=1}^{\infty} \tag{1.104}$$

is an unbounded operator-valued function ($I_\lambda f(1.71) \in D_{M(\lambda)}$).

Note that the Ex. 1.2 demonstrates the possibility of existence of an operator-valued function $M(\lambda)$ with the following properties. It is densely defined and unbounded in $\mathcal{H}(= \mathcal{N}^{\perp})$. However, (1.9) with this $M(\lambda)$ determines on a dense in $L^2_{w_\lambda}(\mathcal{I})$ ($w_\lambda = w_i$) linear manifold a bounded in $L^2_{w_\lambda}(\mathcal{I})$ analytic on λ by [30, p. 195] operator-valued function $R_\lambda = R^*_{\bar{\lambda}}$ which satisfies (1.65).

Remark 1.2. *If one writes down the c.o's $M(\lambda)$ produced in the proof of Th. 1.2, Cases I, II, in the form (1.20), then the corresponding operator-valued function $\mathcal{P}(\lambda)$ is a projection, i.e., $\mathcal{P}^2(\lambda) = \mathcal{P}(\lambda)$.*

Besides that, for those c.o.'s (1.66) is valid even if one replaces in (1.67) P with I.

The proof of the first statement follows from

Lemma 1.9. *Let $P_n^2 = P_n \in B(\mathcal{H})$, $KerP_n = K$ does not depend on n, $P = w - limP_n$. Then $P^2 = P$, $KerP = K$.*

P r o o f o f L e m m a 1.9. Let $f \in K$. Then $Pf = w - limP_n f = 0$, hence $K \subseteq KerP$. Now assume $h \in \mathcal{H}$. Then since $(I - P_n)h \in K$, one has $(I - P)h = w - \lim(I - P_n)h \in K$ due to [30, p. 177]. Therefore if for all $h \in \mathcal{H}$ $P(I - P)h = 0$, then one has $P^2 = P$ and $KerP = K$ since $KerP = (I - P)\mathcal{H} \subseteq K$. Lemma 1.9 is proved.

The second statement follows from the fact that $M_n(\lambda)$ (1.97) is a c.o. for the equation (0.1), (1.92), (1.93) on the interval $(a, \beta_n + 1)$ where $P = I$. Thus for $M_n(\lambda)$ (1.97) one has (1.66) for $\mathcal{I} = (a, \beta_n)$ if $P = I$ in (1.67). Therefore while proving Lem. 1.6 (which in fact gives the desired c.o.) one may set $P = P_n = I$ in (1.90), (1.91), together with the subsequent argument. The remark is proved.

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