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HIGHER MATHEMATICS

GUIDELINES
to independent work

on the topic
"DIFFERENTIAL EQUATIONS"

for Bachelor's (first) degree students

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Guidelines to independent work on the topic "Differential Equations" for Bachelor's (first) degree students reviewed and recommended for publication at the meeting of the Department of Higher Mathematics and physics on 2 December 2024, Minutes No. 4.

The guidelines provide theoretical material and typical examples to help students master the topic of "Differential Equations" and apply the acquired knowledge in practice. They include recommendations for completing independent work assignments, a list of references, theoretical questions, and a self-assessment test to deepen understanding and develop self-study skills.

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Методичні вказівки і завдання для самостійної роботи за темою «Диференціальні рівняння» для студентів першого рівня (бакалаврського) всіх форм навчання розглянуто і рекомендовано до друку на засіданні кафедри вищої математики і фізики 2 грудня 2024 р., протокол № 4.

У методичних вказівках подано теоретичний матеріал і типові приклади, які допоможуть студентам засвоїти тему «Диференціальні рівняння» та застосувати набуті знання на практиці. Також наведено рекомендації для виконання завдань самостійної роботи, список літератури, перелік теоретичних питань і тест для самоперевірки, що сприятиме глибшому розумінню матеріалу та розвитку навичок самостійного навчання.

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Content

Introduction

Mathematical models of many processes and phenomena of various natures (physical, chemical, biological, economic, etc.) are represented by equations that, in addition to functions, include their derivatives.

A *differential equation* (DE) is an equation that contains independent variables, an unknown function, and derivatives (or differentials) of this function.

The differential equation for the function of one independent variable is called an *ordinary differential equation* (ODE). In contrast, *partial differential equation* (PDE) depends on more than one variable.

The ***order*** of the differential equation is the order of the highest derivative in it.

For example,

$yy' - x + 1 = 0$ is the ODE of the first order for unknown function $y(x)$;

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the PDE of the second order for $u(t, x)$.

In these guidelines, we will study ordinary differential equations.

The process of finding an unknown function from a differential equation is called solving or *integrating* a differential equation.

As a result of integrating the ordinary differential equation, the unknown function can be found explicitly or implicitly (i.e., the unknown function is in some relation). We say that in the first case, we find the solution, and in the second – the integral of the differential equation.

In these guidelines, we consider the following topics.

We begin with the study of first-order ordinary differential equations (ODE). The basic definitions of the differential equation and its solution are given. The principal the Cauchy's existence and uniqueness theorem is formulated.

We consider various types of first-order differential equations: ODE with separable variables, homogeneous equations, linear equations, and Bernoulli's equations. We study the methods of solving ODE based on their type.

Then, we study differential equations of higher orders. First, we consider ODEs that allow order reduction by the corresponding substitution. Then, we study methods of integrating second-order linear differential equations with constant coefficients, particularly when the right-hand side has a special form.

At last, we consider systems of linear differential equations with constant coefficients and methods for solving them.

The consideration of each topic is accompanied by a sufficient number of examples. Finally, students are provided with theoretical questions and tests of self-assessment.

1 Ordinary differential equations of the first order: basic concepts

Definition 1. The equation of the form:

$$F(x, y, y') = 0, \tag{1}$$

that contains an independent variable x, an unknown function $y = y(x)$ and its derivative $y' = y'(x)$ is called an *ordinary differential equation (ODE) of the first order*.

First-order differential equation presented in the form

$$y' = f(x, y) \tag{2}$$

solved for the derivative called the *explicit first-order ODE*.

From a geometric point of view, the equation is solved solved for the derivative $y' = f(x, y)$ describes the field of directions of integral curves.

Taking into account that $y' = \frac{dy}{dx}$, equation (2) can also be represented as *the equation in differentials:*

$$P(x,y)dx + Q(x,y)dy = 0\,. \tag{3}$$

The following are examples of differential equations in different forms:

$y'y + tgx = 0$ – the implicit first-order ODE;

$y' = \ln x - y$ – the explicit first-order ODE;

$(2xy - 3)dx + (x^2 + y)dy = 0$ – the equation in differentials.

The process of finding an unknown function of a differential equation is called *solving* or *integrating* a differential equation.

Definition 2. The solution of the differential equation (1) in the interval (a,b) is a function $y = \varphi(x)$ whose substitution into equation (1) tums it into an identity.

If a function, which satisfies equation (1), is found in implicit form, then it is called *the integral* of the differential equation (1).

The graph of the solution function $y = \varphi(x)$ is called *the integral curve*.

Example 1. Consider the differential equation $y' = 2x$.

The function $y = \varphi_1(x) = x^2$ turns it into an identity: $\left(x^2\right)' = 2x$, i.e. it is the solution to this first-order differential equation. However, the functions $\varphi_2(x) = x^2 + 1$, $\varphi_3(x) = x^2 - \sqrt{2}$, as well as any function of the set $\{x^2 + C, \forall C \in \mathbf{R}\}$, are also solutions to this ODE, since

$$\left(x^2\right)' = \left(x^2 + 1\right)' = \left(x^2 - \sqrt{2}\right)' = \left(x^2 + C\right)' = 2x, \quad \forall C \in \mathbf{R}\,.$$

The integral curves of the given equation are parabolas.

Theorem 1 *(Cauchy's existence and uniqueness theorem)*. Let the function $f(x,y)$ and its partial derivative $f'_y(x,y)$ be defined and continuous in the open domain $D \subset \mathbf{R}^2$. Then, for any point $(x_0, y_0) \in D$, there is a unique solution to equation (2) $y = \varphi(x)$ that satisfies the condition:

$$y_0 = \varphi(x_0). \tag{4}$$

Definition 3. Condition (4) is called *the initial condition*, and the problem of finding the solution to the differential equation (2) under condition (4) is called *the Cauchy problem*.

Theorem 1 states that, under certain conditions, there is a unique solution of the Cauchy problem.

From a geometric point of view, Cauchy's theorem states that under certain conditions only one integral curve passes through one point of the plane Oxy.

The points at which the conditions of the theorem are violated are called special. Several curves may pass through them or none at all.

Definition 4. The general solution of a first-order differential equation is a function

$$y = \varphi(x, C), \tag{5}$$

which depends on an arbitrary constant $C \in \mathbf{R}$ and such that:

1) it satisfies the equation (2) for arbitrary real values C;

2) for any given point (x_0, y_0), we can find a value of $C = C_0$ such that the function satisfies the initial condition $\varphi(x_0, C_0) = y_0$.

*Definition 5***.** A *particular solution* of a first-order differential equation is a function $\varphi(x, C_0)$ obtained from the general solution $y = \varphi(x, C)$ of an equation by assigning a specific value $C = C_0$ to the arbitrary constant.

Thus, the Cauchy problem is to find a particular solution that satisfies the given initial condition.

From a geometric point of view, the general solution of the equation describes the set of all integral curves. The particular solution (Cauchy problem) selects one of these curves, namely, the one that passes through a given point (x_0, y_0).

Example 2. Let us solve the Cauchy problem for the differential equation:

$$\begin{cases} y' = 2x \\ y(2) = 3 \end{cases}.$$

The general solution of the equation is of the form:

$$y = x^2 + C, \quad \forall C \in \mathbf{R}.$$

To find the constant corresponding to a particular solution, we substitute the initial condition $y(2) = 3$ into the formula of the general solution:

$$3 = 2^2 + C \qquad \Rightarrow \qquad C = -1.$$

Thus, the solution to the Cauchy problem is the following:

$$y = x^2 - 1.$$

A singular solution of a first-order differential equation is a solution that exists at each point where the uniqueness condition is violated (for example, when the function $f(x, y)$ or its derivative $f'_y(x, y)$ is discontinuous). Singular solutions cannot be obtained from the general solution of the differential equation for any value of an arbitrary constant. The graph of a singular solution is the envelope of the family of integral curves of the differential equation, i.e. a curve that is tangent to at least one integral curve at each point.

Definition 6. If, when solving a differential equation, the general solution is obtained in an implicit form, then the relation $\Phi(x, y, C) = 0$ that defines it is called *the general integral of the equation*. The relation obtained from the general integral for a specific value of the parameter $C = C_0$: $\Phi(x, y, C_0) = 0$, is called *the particular integral of the equation*.

Next, we turn to the study of different types of differential equations and the corresponding methods for solving them.

2 The simplest differential equations

Definition 7. Equations of the form

$$y' = f(x) \tag{6}$$

are called *the simplest first-order differential equations*.

To find a general solution, one multiplies both sides of the equality by dx and integrates:

$$\int y'dx = \int f(x)dx;$$

$$y = \int f(x)dx + C.$$

Example 2. Solve the equation: $y' = \cos 2x$.

Solution. Indefinite integration gives the general solution of the equation:

$$\int y'dx = \int \cos 2x dx;$$

$$y = \frac{\sin 2x}{2} + C.$$

Substituting the found solution into the given equation proves the correctness of calculations:

$$\left(\frac{\sin 2x}{2} + C \right)' \equiv \cos 2x, \ \forall x \in R$$

Answer: $y = \frac{\sin 2x}{2} + C$.

3 Differential equations with separable variables

Definition 8. A first-order differential equation of the form

$$y' = f(x) \cdot g(y) \qquad (7)$$

is called *differential equations with separable variables.*

To integrate such equations, we first need to express the derivative as a fraction of differentials:

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Next, we need to transform the equation so that the left-hand side depends only on y, and the right-hand side depends only on x:

$$\frac{dy}{g(y)} = f(x)dx, \ (g(y) \neq 0).$$

Further integration of the left-hand side of the equality by the variable y and the the right-hand side by the variable x will allow obtaining the general integral (implicit solution) of the original solution:

$$\int \frac{dy}{g(y)} = \int f(x)dx + C.$$

The condition $g(y) = 0$ is considered separately. There are three possible cases:

- $g(y) = 0$ does not give a solution;
- this condition corresponds to a particular solution;
- the condition provides a singular solution.

Example 3. Solve the equation: $y' = \frac{y}{x}$.

Solution. We have a differential equation with separable variables. We write the derivative as the fraction of differentials:

$$\frac{dy}{dx} = \frac{y}{x}.$$

Divide the equation by $y \neq 0$ and multiply by dx to separate the variables:

$$\frac{dy}{y} = \frac{dx}{x} \quad (y \neq 0).$$

Now we integrate both sides over the corresponding variables:

$$\int \frac{dy}{y} = \int \frac{dx}{x};$$

$$\ln|y| = \ln|x| + \ln|C| \quad (C \neq 0).$$

In the last equation, for convenience of further transformations, an arbitrary constant is written in the form $\ln|C|\,(C \neq 0)$. From the last equality we obtain the general solution of the given differential equation:

$$y = Cx \quad (C \neq 0).$$

Then we check the function $y = 0$ and establish that it satisfies the given equation. We see that this particular solution can be added to the general one by rejecting the condition $C \neq 0$. Finally, the set of solutions of the original equation is as follows:

$$y = Cx, \quad \forall C \in \mathbf{R}.$$

Answer: $y = Cx, \quad \forall C \in \mathbf{R}$.

Remark 1. The differential equations of the form

$$y' = f(ax + by + c)$$

can be reduced to an equation with separable variables by replacing of the unknown function $y(x)$ with

$$u = ax + by + c;$$

$$\text{then} \qquad u' = a + by' \quad \Leftrightarrow \quad y' = \frac{1}{b}(u' - a).$$

4 First-order homogeneous differential equations

Definition 9. The function $f(x, y)$ is called homogeneous of degree k, if for any $\lambda \neq 0$ the following equality is holds:

$$f(\lambda x, \lambda y) = \lambda^k f(x, y).$$

Definition 10. First-order homogeneous differential equations are equations of the form

$$y' = f(x, y),$$

where $f(x, y)$ is a continuous homogeneous function: $f(\lambda x, \lambda y) = f(x, y)$.

In addition to the definition, the following criteria can be used to identify the homogeneity of the ODE:

1 A homogeneous equation can be represented as

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

2 All terms of a homogeneous differential equation have the same degree in the totality of variables x and y.

For example, in the differential equation

$$xy' = y + \sqrt{xy},$$

the degrees of all components in the totality of variables x and y are the same and equal to 1, and on the other hand, it can be represented as

$$y' = \frac{y}{x} + \sqrt{\frac{y}{x}},$$

where the right-hand side is a function of the form $f\left(\frac{y}{x}\right)$, and therefore homogeneous:

$$f(\lambda x,\ \lambda y) = \frac{\lambda y}{\lambda x} + \sqrt{\frac{\lambda y}{\lambda x}} = \frac{y}{x} + \sqrt{\frac{y}{x}} = f(x,\ y).$$

A homogeneous differential equation is reduced to an equation with separable variables by replacing an unknown function $y = y(x)$ with $u = u(x)$:

$$u = \frac{y}{x} \ \Rightarrow\ y = ux,\ y' = u'x + u. \qquad (9)$$

Indeed, equation (8) takes the form:

$$u'x + u = f(u),$$

which is an equation with separable variables for a function $u = u(x)$:

$$u' = \frac{f(u) - u}{x}.$$

Example 4. Let us solve the equation: $xyy' = x^2 + y^2$.

Solution. Since all components of the equation have the first degree in the totality of variables x and y, it is a first-order homogeneous differential equation.

On the other hand, if we represent this equation in explicit form (solve for the derivative of the unknown function), we obtain the equation of the form (8):

$$y' = \frac{x}{y} + \frac{y}{x}, \ (xy \neq 0).$$

Finally, the right-hand side of this equation is obviously a homogeneous function:

$$f(\lambda x, \ \lambda y) = \frac{\lambda x}{\lambda y} + \frac{\lambda y}{\lambda x} = \frac{x}{y} + \frac{y}{x} = f(x, \ y).$$

Since the given equation is homogeneous, we use the substitution (9) and obtain:

$$u = \frac{y}{x}, \ y' = u'x + u \ \Rightarrow \ u'x + u = \frac{1}{u} + u;$$

$$u'x = \frac{1}{u}.$$

We have an equation with separable variables, so we apply the standard procedure: we represent the derivative as a fraction of differentials $u' = \frac{du}{dx}$ and separate variables:

$$\frac{du}{dx}x = \frac{1}{u};$$

$$udu = \frac{dx}{x}.$$

Now we integrate both sides of the equation to obtain the general integral for the function u:

$$\int u du = \int \frac{dx}{x};$$

$$\frac{u^2}{2} = \ln|x| + C_1.$$

We return to the unknown function $y = ux$ and obtain the general integral of the original equation:

$$y^2 = x^2\left(2\ln|x| + C\right), \ (C = 2C_1).$$

The condition $xy = 0$ does not give singular solutions.

Answer: $y^2 = x^2\left(2\ln|x| + C\right)$.

5 Linear differential equations of the first order

Definition 11. A linear inhomogeneous differential equation of the first order is an equation that is linear with respect to an unknown function and its derivative:

$$y' + p(x)y = g(x), \tag{10}$$

where $p(x)$ and $g(x) \neq 0$ are given continuous on some interval functions of x.

Definition 12. If $g(x) \equiv 0$, then an equation

$$y' + p(x)y = 0 \tag{11}$$

is called *a linear homogeneous differential equation*.

(11) is an equation with separable variables.

We consider two methods of integrating equations (10).

1) Bernoulli's method (the method of auxiliary functions). The essence of the method is that the solution of the equation is sought in the form of the product of two functions of x:

$$y = u(x) \cdot v(x) .$$

Then the derivative y' is:

$$y' = u'v + uv',$$

and after substituting the expressions for y and y' in the original equation (10) we obtain:

$$u'v + (v' + p(x)v)u = g(x) . \tag{12}$$

The function $v(x)$ we choose so that the factor at $u(x)$ in equation (12) becomes zero, then according to it determine the function $u(x)$. Thus, we have a system of equations for finding $u(x)$ and $v(x)$:

$$\begin{cases} v' + p(x) \cdot v = 0, \\ u' \cdot v = g(x), \end{cases} \tag{13}$$

whose equations are differential equations with separable variables, that are two steps of solving (10).

First, we integrate the first equation of the system (13) and take the simplest function $v(x)$ that satisfies it:

$$\frac{dv}{dx} + p(x)v = 0;$$

$$\frac{dv}{v} = -p(x)dx;$$

$$\ln|v| = -\int p(x)\,dx;$$

$$v(x) = e^{-\int p(x)\,dx}.$$

Next, we substitute the found solution $v(x)$ in the second equation of the system (13) and find the function $u(x)$:

$$\frac{du}{dx} = g(x)e^{\int p(x)\,dx};$$

$$du = g(x)e^{\int p(x)\,dx}\,dx;$$

$$u(x) = \int g(x)e^{\int p(x)\,dx}\,dx + C.$$

Finally, the general solution of linear equation (10) has the form:

$$y(x) = u(x)v(x) = e^{-\int p(x)dx}\left(\int g(x)e^{\int p(x)dx}\,dx + C\right). \qquad (14)$$

Example 5. Solve the linear inhomogeneous differential equation of the first order: $y' - 2y = e^{3x}$.

Solution. We assume that $y = u \cdot v$ then $y' = u'v + uv'$ and the given equation becomes:

$$u'v + uv' - 2uv = e^{3x};$$

$$u'v + u(v' - 2v) = e^{3x}.$$

Let $v' - 2v = 0$, then $u'v = e^{3x}$, which means the solution of the original equation reduces to the solution of the system:

$$\begin{cases} v' - 2v = 0; \\ u'v = e^{3x}. \end{cases}$$

First, we find the solution of the first equation $v' - 2v = 0;$

$$\frac{dv}{dx} = 2v;$$

$$\frac{dv}{v} = 2dx;$$

$$\int \frac{dv}{v} = 2\int dx;$$

$$\ln|v| = 2x + \ln|C|;\ (C \neq 0)$$

$$v = e^{2x}.$$

Now we substitute the found function $v(x)$ into the second equation and solve it for the function $u(x)$:

$$u'v = e^{3x};$$

$$u'e^{2x} = e^{3x};$$

$$u' = e^{x};$$

$$\int u'dx = \int e^{x}dx;$$

$$u = e^x + C.$$

Thus, the general solution of the given linear inhomogeneous differential equation of the first order has the form:

$$y = uv = e^{2x}\left(e^x + C\right) = e^{3x} + Ce^{2x}.$$

Let us make a substitution check:

$$\left(e^{3x} + Ce^{2x}\right)' - 2\left(e^{3x} + Ce^{2x}\right) = e^{3x};$$

$$3e^{3x} + 2Ce^{2x} - 2e^{3x} - 2Ce^{2x} = e^{3x};$$

$$e^{3x} \equiv e^{3x}.$$

We have verified that the obtained general solution $y = e^{3x} + Ce^{2x}$ turns the equation into an identity.

Answer: $y = e^{3x} + Ce^{2x}$.

2) Lagrange's method (method of variation of an arbitrary constant).

The solution of the first-order linear inhomogeneous first-order differential equation (10) using this method is also divided into two stages.

First, we find the general solution of a linear homogeneous differential equation of the first order (11), which we obtain from (10) by setting the right-hand side to zero:

$$y' + p(x)y = 0;$$

$$\frac{dy}{y} = -p(x)dx;$$

$$\ln|y| = -\int p(x)dx + \ln|C|;$$

$$y = Ce^{-\int p(x)dx},\ C \neq 0. \qquad (15)$$

Now let us substitute into the found general solution (15) of the homogeneous equation (11) instead of the parameter C the unknown differentiable function $C = C(x)$, i.e. we will look for the general solution of the linear inhomogeneous equation (10) in the form

$$y = C(x)e^{-\int p(x)dx}. \qquad (16)$$

We differentiate (16)

$$y' = C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx}$$

and substitute y and y' into the initial equation (10):

$$C'(x)e^{-\int P(x)dx} - C(x)p(x)e^{-\int p(x)dx} + p(x)C(x)e^{-\int p(x)dx} = g(x);$$

$$C'(x) = g(x)e^{\int p(x)dx}.$$

By integrating the last differential equation, we find the function $C(x)$:

$$C(x) = \int g(x)e^{\int p(x)dx}dx + C.$$

Thus, the general solution of the linear inhomogeneous equation has the form:

$$y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int g(x)e^{\int p(x)dx}dx,$$

where the first term is the general solution of the linear inhomogeneous equation, and the second term is the particular solution of the linear inhomogeneous equation.

It is easy to see that the results obtained by two methods coincide.

Example 6. Solve the linear inhomogeneous differential equation of the first order from the previous example using the Lagrange method: $y' - 2y = e^{3x}$.

Solution. First, we find a general solution of the corresponding linear homogeneous first-order differential equation:

$$y' - 2y = 0;$$

$$\frac{dy}{y} = -2dx;$$

$$\ln|y| = 2\int dx + \ln|C|;$$

$$y = Ce^{2x}.$$

Now we put $C = C(x)$ in the found general solution, i.e. we will look for the general solution of the linear inhomogeneous equation in the form:

$$y = C(x)e^{2x}.$$

The derivative of this function is equal to:

$$y' = C'(x)e^{2x} + 2C(x)e^{2x}.$$

Substitute y and y' in a given equation (10):

$$C'(x)e^{2x} + 2C(x)e^{2x} - 2C(x)e^{2x} = e^{3x};$$

$$C'(x)e^{2x} = e^{3x};$$

$$C'(x) = e^{x}.$$

Integrating the last differential equation, we find the function $C(x)$:

$$\int C'(x)dx = \int e^{x}dx;$$

$$C(x) = e^{x} + C.$$

Substitute $C(x)$ in (18) and we obtain the general solution of the original linear inhomogeneous equation:

$$y = e^{3x} + Ce^{2x}.$$

Answer: $y = e^{3x} + Ce^{2x}$.

Remark 2. Differential equation of the form

$$y' + p(x)\,y = y^{n} g(x), \text{ where } n \in \mathbf{R} \setminus \{0;1\}, \tag{17}$$

is called *Bernoulli's equation*.

Here $p(x)$ and $g(x)$ are given integrable functions.

The Bernoulli equation is a non-linear equation, since the unknown function $y(x)$ on the right-hand side of the equation has a non-unit power. For $n > 0$ the function $y = 0$ is a particular solution of (17); when $n = 0$ or $n = 1$ it is linear with respect to unknown function.

One can solve Bernoulli's equation using the aforementioned Bernoulli's method, i.e. by looking for an unknown function in the form $y = u(x)\cdot v(x)$.

The second way is to reducing (17) to a first-order linear equation by replacing the unknown function $y(x)$ with the function $z(x)$ according to the formula:

$$z(x) = y^{1-n};\ y' = \frac{1}{1-n} y^n z'.$$

6 Higher order differential equations: basic concepts

Definition 13. An ordinary differential equation of order n is the equation:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{18}$$

that relates an independent variable x, an unknown function $y = y(x)$, and its derivatives $y', y'', \ldots, y^{(n)}$.

Definition 14. A differential equation

$$y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right) \tag{19}$$

resolved with respect to the highest derivative is called the *explicit ODE of order* n.

Definition 15. An n times differentiable function $y = \varphi(x)$ on a set D, the substitution of which into ODE (18) turns it into an identity is called *a solution* of n-th order differential equation on D.

Definition 16. The Cauchy problem for n-th order ODE is to find a solution of the equation that satisfies the following n initial conditions:

$$y(x_0) = y_0,\ y'(x_0) = y'_0,\ ...,\ y^{(n-1)}(x_0) = y_0^{(n-1)}. \qquad (20)$$

Definition 17. The general solution of a differential equation of the n-th order is a function

$$y = \varphi\left(x, C_1, C_2, ..., C_n\right), \qquad (21)$$

which depends on n arbitrary constants $C_1, C_2, ..., C_n$, $C_i \in \mathbf{R}$, $i = \overline{1,n}$ and satisfies the following conditions:

1) the function $y = \varphi\left(x, C_1, C_2, ..., C_n\right)$ is the solution of the differential equation for any constant values $C_1, C_2, ..., C_n$;

2) for arbitrary initial conditions (20) there are such constants $C_1^0, C_2^0, ..., C_n^0$ under which the solution $y = \varphi\left(x, C_1^0, C_2^0, ..., C_n^0\right)$ satisfies the initial conditions.

Definition 18. A particular solution of the n-th order differential equation is a function

$$y = \varphi\left(x, C_1^0, C_2^0, ..., C_n^0\right), \qquad (22)$$

that is obtained from the general solution $y = \varphi\left(x, C_1, C_2, ..., C_n\right)$ at certain values of the constants $C_i = C_i^0$, $i = \overline{1,n}$.

Definition 19. If the general solution of the n-th order differential equation is found in an implicit form, i.e., in the form of an equation $\Phi\left(x, y, C_1, C_2, ..., C_n\right) = 0$, then such a solution is called the *general integral* of the differential equation. An implicit form of a particular solution is called a *particular integral* of a differential equation.

Theorem 2 *(Cauchy's existence and uniqueness theorem)*. If in the n-th order differential equation (19) the function $f\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$ is continuous and has continuous partial derivatives with respect to variables $y, y', y'', \ldots, y^{(n-1)}$ on a set $\boldsymbol{D} \in \mathbf{R}^{\boldsymbol{n}+1}$, that contains the point $\left(x_0,\, y_0,\, y_0', \ldots,\, y_0^{(n-1)}\right)$, then there is a unique solution to this equation that satisfies the given initial conditions (20).

Note that in contrast to the first-order differential equation, through each point in space $\mathbf{R}^{\boldsymbol{n}}$ $(n \geq 2)$ not just one, but many integral curves of the n-th order differential equation pass.

In the particular case $n = 2$, a second-order differential equation has the form

$$F\left(x, y, y', y''\right) = 0, \tag{23}$$

the explicit form of the second-order differential equation is

$$y'' = f\left(x, y, y'\right), \tag{24}$$

and the initial condition has the form

$$y(x_0) = y_0,\ y'(x_0) = y_0'. \tag{25}$$

The general solution of the second-order differential equation is a function that depends on two arbitrary constants:

$$y = \varphi\left(x, C_1, C_2\right),\ C_1, C_2 \in \mathbf{R}.$$

7 Differential equations of higher orders that allow order reduction

Consider the simplest cases of incomplete differential equations the order of which can be reduced.

1 ***The equation contains only the independent variable*** x ***and the highest derivative*** $y^{(n)}$***:***

$$y^{(n)} = f(x). \tag{26}$$

The general solution of such equation can be found by multiple integrations of the original equation.

Example 7. Solve the equation: $y''' = \sin 2x$.

Solution. Integrate both parts of this equation three times and get:

$$\int y''' dx = \int \sin 2x dx;$$

$$y'' = -\frac{1}{2}\cos 2x + C_1;$$

$$\int y'' dx = -\frac{1}{2}\int \cos 2x dx + \int C_1 dx;$$

$$y' = -\frac{1}{4}\sin 2x + C_1 x + C_2;$$

$$\int y' dx = -\frac{1}{4}\int \sin 2x dx + \int C_1 x dx + \int C_2 dx.$$

Finally, we obtain a general solution of the original third-order differential equation:

$$y = \frac{1}{8}\cos 2x + C_1\frac{x^2}{2} + C_2 x + C_3.$$

Answer: $y = \frac{1}{8}\cos 2x + C_1 \frac{x^2}{2} + C_2 x + C_3.$

2 ***Second-order differential equations that do not contain the explicitly sought function*** y***:***

$$F\left(x, y', y''\right) = 0. \tag{27}$$

Changing the original unknown function $y(x)$ to $u(x)$, defined by the formula

$$u(x) = y'(x) \text{, then } u'(x) = y''(x), \tag{28}$$

we reduce (27) to the first-order ODE for the function $u(x)$:

$$F\left(x, u, u'\right) = 0.$$

3 ***Second-order differential equations that do not contain the explicitly independent variable*** x***:***

$$F\left(y, y', y''\right) = 0. \tag{29}$$

Changing an unknown function $y(x)$ with $u(y)$ by the formula

$$y' = u(y),\ y'' = u'(y)y' = u'(y)u(y) \tag{30}$$

we reduce (27) to the first-order ODE for the function $u(y)$:

$$F(y,u,u')=0.$$

Example 8. Solve the equation: $y''=y'$.

Solution. Since the equation contains neither an the independent variable x nor an unknown function y explicitly, it is both an equation of types (27) and (29), so it can be solved in two ways.

1st method. Use the substitution (28), i.e. let $u=y'$, then $u'=y''$ and the given equation takes the form:

$$u'=u\ .$$

Solve the resulting equation with separable variables:

$$\frac{du}{dx}=u;$$

$$\frac{du}{u}=dx;$$

$$\ln|u|=x+\ln|C_1|;$$

$$u=C_1e^x.$$

Perform the back-substitution $u=y'$, obtaining the first-order ODE for the original function $y(x)$:

$$y'=C_1e^x$$

and solve this equation by separation of variables to find $y(x)$:

$$\frac{dy}{dx} = C_1 e^x;$$

$$dy = C_1 e^x dx;$$

$$\int dy = \int C_1 e^x dx.$$

Thus, the general solution of the original equation has the form:

$$y = C_1 e^x + C_2, \ C_1, C_2 \in \mathbf{R}.$$

2nd method. According to (30), we assume: $u = y'$, then $u'u = y''$. Substitution of these expressions leads to the simplest equation for the function $u(y)$:

$$u'u = u \ \Rightarrow \ u' = 1.$$

Integration with respect to y gives us the general solution:

$$\int u' dy = \int dy;$$

$$u = y + C_1.$$

Now we return to the original variable, substituting $u = y'$:

$$y' = y + C_1$$

and solve the equation with separable variables for the function $y(x)$:

$$\frac{dy}{dx} = y + C_1;$$

$$\frac{dy}{y+C_1}=dx;$$

$$\ln|y+C_1|=x+\ln|C_2|.$$

Hence, the general solution of the original equation is:

$$y=C_2e^x+C_1,\ C_1,C_2\in\mathbf{R}.$$

Comparing the results, we see that the general solutions found by different methods coincide up to the notation of constants.

Answer: $y=C_2e^x+C_1$, $C_1,C_2\in\mathbf{R}$.

8 Homogeneous linear differential equations of the second order with constant coefficients

Definition 20. A linear differential equation of the n-th order with constant coefficients is an equation of the form

$$y^{(n)}+a_1y^{(n-1)}+...+a_ny=f(x), \tag{31}$$

where a_1, a_2, …, a_n are real numbers.

The term "linear" is related to the fact that the equation contains an unknown function $y(x)$ and all its derivatives only in the first degree.

We will focus on the study of the particular case $n=2$.

Definition 21. An inhomogeneous linear second-order differential equation with constant coefficients is an equation of the form

$$y''+py'+qy=f(x), \tag{32}$$

where coefficients $p, q \in \mathbf{R}$.

Definition 22. A homogeneous linear second-order differential equation is an equation with zero on the right-hand side:

$$y'' + py' + qy = 0, \qquad p, q \in R. \tag{33}$$

Consider the properties of the solutions of the equation (33).

Theorem 3. If a function $y = y_1(x)$ is a solution of equation (33), then any function $y = Cy_1(x)$, where C is an arbitrary constant is also a solution of this equation.

Theorem 4. If the functions $y_1(x)$ and $y_2(x)$ are the solutions of equation (33), then their sum $y = y_1(x) + y_2(x)$ is also the solution of this equation.

Definition 23. Two nonzero solutions $y_1(x)$ and $y_2(x)$ are called *linearly dependent* in the interval (a,b) if their ratio is equal to a constant number: $\frac{y_1}{y_2} = \lambda$, $\lambda - \text{const}$. Otherwise, the solutions $y_1(x)$ and $y_2(x)$ are called *linearly independent*: $\frac{y_1}{y_2} \neq \lambda$, where $\lambda - \text{const}$.

Definition 24. A set $\{y_1(x), y_2(x)\}$ of linearly independent in the interval (a,b) solutions of (33) is called a *fundamental system of solutions for* this differential equation.

Definition 25. The Wronskian of two differentiable functions $y_1(x)$, $y_2(x)$ is a second-order determinant whose elements are the functions themselves and their derivatives:

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2. \tag{34}$$

Theorem 5. Solutions $y_1(x)$, $y_2(x)$ of the equation (33) form a fundamental system of solutions if and only if $W(y_1, y_2) \neq 0$ on (a, b).

Theorem 6 *(on the structure of the general solution of a homogeneous linear second-order differential equation).* If $\{y_1(x), y_2(x)\}$ is the fundamental system of solutions of equation (33), then its general solution has the form

$$y_o(x) = C_1 y_1(x) + C_2 y_2(x), \tag{35}$$

where C_1, C_2 are arbitrary constants.

Thus, the general solution of equation (33) is a linear combination of the functions of a fundamental system of solutions.

Definition 26. The *characteristic equation* of a homogeneous linear second-order differential equation with constant coefficients

$$y'' + py' + qy = 0, \quad p, q \in \mathbf{R}$$

is the following quadratic equation for a numerical parameter k

$$k^2 + pk + q = 0. \tag{36}$$

The fundamental system of solutions of a homogeneous linear equation (33) and, as a consequence, its general solution depends on the form of solutions of the characteristic equation (36).

There are 3 cases:

1) roots of the characteristic equation (36) are real and different $(D > 0)$; in this case, a fundamental system of solutions is $\left\{e^{k_1 x}, e^{k_2 x}; k_1, k_2 \in \mathbf{R}; k_1 \neq k_2\right\}$;

2) (36) has a single real root with multiplicity 2 $(D=0)$; then a fundamental system of solutions is $\left\{e^{kx}, xe^{kx}; k_1=k_2=k\in\mathbf{R}\right\}$;

3) (36) has complex conjugate roots $(D<0)$, a fundamental system of solutions takes the form $\left\{e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x; k_{1,2}=\alpha\pm i\beta\in\mathbf{C}\right\}$.

Table 1 illustrates the correspondence between the roots of the characteristic equation (36) and the general solution of the homogeneous linear equation with constant coefficients (33).

Table 1 – General solution of a homogeneous linear equation

№	Roots of the characteristic equation $k^2+pk+q=0$	General solution of the equation $y''+py'+qy=0$
1	$k_1, k_2\in\mathbf{R}$; $k_1\neq k_2$	$y_o=C_1e^{k_1x}+C_2e^{k_2x}$, $\forall C_1,C_2\in\mathbf{R}$
2	$k_1=k_2=k\in\mathbf{R}$	$y_o=(C_1+C_2x)\,e^{kx}$, $\forall C_1,C_2\in\mathbf{R}$
3	$k_1, k_2\in\mathbf{C}$; $k_{1,2}=\alpha\pm i\beta$	$y_o=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$, $\forall C_1,C_2\in\mathbf{R}$

Example 9. Find the general solution to the homogeneous linear equations with constant coefficients:

a) $y''-5y'+6y=0$;

b) $y''+6y'+9y=0$;

c) $y''-4y'+13y=0$.

Solution.

a) For the given equation $y''-5y'+6y=0$ compose a characteristic equation

$$k^2 - 5k + 6 = 0$$

and find its roots using Vieta's theorem:

$$k_1 = 2, k_2 = 3.$$

Since the roots are real and different, we have the first case. According to Table 1, we write the general solution:

$$y_o = C_1 e^{2x} + C_2 e^{3x}, \qquad C_1, C_2 \in \mathbf{R}.$$

b) From the equation $y'' + 6y' + 9y = 0$ compose a characteristic equation and find its roots:

$$k^2 + 6k + 9 = 0;$$

$$(k+3)^2 = 0;$$

$$k_1 = k_2 = k = -3.$$

We have a single root of multiplicity 2, so the general solution has the form:

$$y_o = C_1 e^{-3x} + C_2 x e^{-3x}, \qquad C_1, C_2 \in \mathbf{R}.$$

c) The characteristic equation for $y'' - 4y' + 13y = 0$ is of the form:

$$k^2 - 4k + 13 = 0.$$

The discriminant of this quadratic equation is negative:

$$D = 4^2 - 4 \cdot 13 = -36$$

and the roots are complex conjugates:

$$k_{1,2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i .$$

According to Table 1, we write the general solution of the given equation:

$$y_o = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right), \quad C_1, C_2 \in \mathbf{R}.$$

Answer: a) $y_o = C_1 e^{2x} + C_2 e^{3x}, C_1, C_2 \in \mathbf{R}$;

b) $y_o = C_1 e^{-3x} + C_2 x e^{-3x}, C_1, C_2 \in \mathbf{R}$;

c) $y_o = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right), C_1, C_2 \in \mathbf{R}$.

9 Inhomogeneous linear differential equations of the second order with constant coefficients and the right part of the special form

In this section, we study the properties of solutions of inhomogeneous linear equations (32) with a special right-hand side and the methods of their solution.

Theorem 7 *(on the structure of the general solution of an inhomogeneous linear second-order differential equation*). The general solution of an inhomogeneous linear differential equation of the second order is equal to the sum of the general solution $y_o(x)$ of the corresponding homogeneous equation (33) and any particular solution $\tilde{y}(x)$ of this inhomogeneous equation (32):

$$y(x) = y_o(x) + \tilde{y}(x) . \qquad (37)$$

Definition 27. An *inhomogeneous linear equation with a special right-hand side* is an equation (32), whose right-hand side has the form:

$$f(x) = e^{ax}(P_n(x)\cos bx + Q_m(x)\sin bx), \qquad (38)$$

where a, b are real constants; $P_n(x)$, $Q_m(x)$ are polynomials of degree n and m respectively.

According to the structure of the general solution of the inhomogeneous equation, the problem is reduced to finding a particular solution of the equation $\tilde{y}(x)$, the form of which is determined by the values of the numerical parameters a, b, n, m of the right-hand side (38).

Theorem 8 *(on the particular solution of an inhomogeneous linear second-order equation with a special right-hand side*). If the inhomogeneous equation (32) has a special right-hand side (38):

$$f(x) = e^{ax}(P_n(x)\cos bx + Q_m(x)\sin bx),$$

then its particular solution $\tilde{y}(x)$ can be found in the form:

$$\tilde{y}(x) = x^r e^{ax}(U_k(x)\cos bx + V_k(x)\sin bx), \qquad (39)$$

where $U_k(x)$, $V_k(x)$ are polynomials of degree $k = \max\{n, m\}$, and parameter the r shows how many roots of the characteristic equation coincide with a number $a + bi$, $i = \sqrt{-1}$.

Remark 3. The number r can take one of three values:

$$r = 0, \text{ if } a + bi \neq k_1 \neq k_1 \text{ and } a + bi \neq k_2;$$

$$r = 1, \text{ if } a + bi = k_1 \text{ or } a + bi = k_2;$$
$$r = 2, \text{ if } a + bi = k_1 = k_2 = k,$$

where k_1, k_2 are roots of the characteristic equation.

Remark 4. To find a particular solution to inhomogeneous equation (32), we must compose it by formula (39) with polynomials having undetermined coefficients, and then select them so that the function (39) satisfies (32). The following are examples of polynomials of degrees 0, 1, 2 and 3 with undetermined coefficients:

$$U_0(x) = A;$$
$$U_1(x) = Ax + B;$$
$$U_2(x) = Ax^2 + Bx + C;$$
$$U_3(x) = Ax^3 + Bx^2 + Cx + D.$$

Some specific cases of the right-hand sides of equation (38) and the corresponding type of particular solutions are given in Table 2.

Table 2 – Particular solutions of a linear inhomogeneous equations

№	Right-hand side $f(x) = e^{ax}(P_n(x)\cos bx + Q_m(x)\sin bx)$	Particular solutions $\tilde{y}(x) = x^r e^{ax}(U_k(x)\cos bx + V_k(x)\sin bx)$
1	$f(x) = P_n(x)$, $(a = b = 0)$	$\tilde{y} = x^r U_n(x)$
2	$f(x) = P_n(x)e^{ax}$, $(b = 0)$	$\tilde{y} = x^r e^{ax} U_n(x)$
3	$f(x) = P_n(x)\cos bx + Q_m(x)\sin bx$, $(a = 0)$	$\tilde{y} = x^r(U_k(x)\cos bx + V_k(x)\sin bx)$

Theorem 9 *(on the superposition of solutions of an inhomogeneous linear differential equation).* If the right-hand side of the inhomogeneous differential equation (32) is the sum of two special functions $f(x) = f_1(x) + f_2(x)$, then the particular solution of this equation is defined as the sum of corresponding particular solutions: $\tilde{y} = \tilde{y}_1 + \tilde{y}_2$.

Example 10. Find the general solution to the equation: $y'' - 6y' + 9y = 3x - 8e^x$.

Solution.

First, we solve the corresponding homogeneous equation $y'' - 6y' + 9y = 0$. To do this, we form the characteristic equation: $k^2 - 6k + 9 = 0$ and find its roots: $k_1 = k_2 = 3$. Since the root is repeated, the general solution of the homogeneous equation has the form

$$y_o = C_1 e^{3x} + C_2 x e^{3x}.$$

The right-hand side of the given equation is the sum of two special functions $f_1(x) = 3x$ and $f_2(x) = -8e^x$.

We analyze $f_1(x) = 3x$, find the values of the parameters a, b, n, m and determine the type of $\tilde{y}_1(x)$:

$$\left[\begin{array}{l}(a = b = 0,\ n = 1) \Rightarrow U_1(x) = Ax + B, \\ (a + bi = 0 \neq k = 3) \Rightarrow r = 0\end{array}\right. \Rightarrow \tilde{y}_1(x) = x^r U_1(x) = Ax + B.$$

Similarly, for $f_2(x) = -8e^x$ we have:

$$\left[\begin{array}{l}(a = 1,\ b = 0,\ n = 0) \Rightarrow U_0(x) = C, \\ (a + bi = 1 \neq k = 3) \Rightarrow r = 0\end{array}\right. \Rightarrow \tilde{y}_2(x) = x^r e^x U_0(x) = Ce^x.$$

A particular solution of the equation $\tilde{y}(x)$ is the sum of two functions $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$:

$$\tilde{y} = \tilde{y}_1 + \tilde{y}_2 \Rightarrow \tilde{y} = Ax + B + Ce^x.$$

Find the first and second derivative functions and substitute them, along with the function $\tilde{y}(x)$ itself, into the original equation:

$$Ce^x - 6(A + Ce^x) + 9(Ax + B + Ce^x) = 3x - 8e^x.$$

We equate the coefficients of e^x and equal powers of x in both parts of the equation and obtain a system of linear equations with respect to unknown parameters:

$$\begin{array}{c|l} e^x & \\ x^1 & \\ x^0 & \end{array} \begin{cases} 4C = -8, \\ 9A = 3. \\ -6A + 9B = 0. \end{cases}$$

As a result of solving this system, we obtain the values of coefficients:

$$A = 1/3,\ B = 2/9,\ C = -2.$$

Hence, a particular solution of the inhomogeneous equation has the form

$$\tilde{y} = 1/3x + 2/9 - 2e^x.$$

Thus, according to (37) a general solution of the given equation is

$$y = C_1 e^{3x} + C_2 xe^{3x} + \frac{x}{3} + \frac{2}{9} - 2e^x.$$

Answer: $y = C_1 e^{3x} + C_2 xe^{3x} + \frac{x}{3} + \frac{2}{9} - 2e^x$.

10 Solving inhomogeneous linear differential equations by the method of variation of arbitrary constants

The inhomogeneous linear differential equation (31) can be solved using the method of variation of arbitrary constants, regardless of the form of the right side.

The application of this method is similar to solving first-order linear differential equations (see section 5). We will consider the implementation of the method of variation of arbitrary constants for inhomogeneous second-order linear differential equations (32).

First, for a homogeneous linear second-order differential equation

$$y'' + py' + qy = 0, \quad p, q \in \mathbf{R},$$

which we obtain from (32) by equating to zero the right-hand side, we find a general solution as a linear combination of functions of the fundamental system of solutions:

$$y = C_1 y_1(x) + C_2 y_2(x). \qquad (40)$$

Now, in the particular solution (40) of the homogeneous equation found, we set the parameters C_1, C_2 equal to the unknown differentiable functions

$C_i = C_i(x), i = 1,2$, i.e. we will look for the particular solution of the linear inhomogeneous equation (32) in the form

$$y = C_1(x)\,y_1(x) + C_2(x)\,y_2(x)\ , \tag{41}$$

where the unknown functions $C_1(x), C_2(x)$ are determined from a system of equations:

$$\begin{cases} C_1'(x)\,y_1(x) + C_2'(x)\,y_2(x) = 0, \\ C_1'(x)\,y_1'(x) + C_2'(x)\,y_2'(x) = f(x). \end{cases} \tag{42}$$

Next, we solve a linear system (42) with respect to the derivatives of the unknown functions $C_i' = C_i'(x)$, $i = 1,2$.

Finally, we find the functions $C_i = C_i(x), i = 1,2$ by indefinite integration:

$$C_1(x) = -\int \frac{y_2(x)\,f(x)}{W(x)}dx,\ C_2(x) = \int \frac{y_1(x)\,f(x)}{W(x)}dx, \tag{43}$$

where $W(x) = W(y_1, y_2)$ is Wronskian of functions of the fundamental system of solutions. Substituting $C_1(x), C_2(x)$ in (41) we obtain a particular solution of the given inhomogeneous equation.

Example 11. Find the general solution of the equation: $y'' - 2y' - 3y = \dfrac{e^{3x}}{9 + e^{2x}}$.

Solution.

First, consider the corresponding homogeneous differential equation with constant coefficients:

$$y'' - 2y' - 3y = 0.$$

Its characteristic equation $k^2 - 2k - 3 = 0$ has the roots $k_1 = -1,\ \ k_2 = 3$. Hence, according to Table 1 the general solution of the homogeneous equation is as follows:

$$y_o = C_1 e^{-x} + C_2 e^{3x}$$

where $y_1 = e^{-x}$, $y_2 = e^{3x}$ are its linearly independent solutions.

According to the method of variation of arbitrary constants, a particular solution of the inhomogeneous equation is sought in the form:

$$\tilde{y} = C_1(x)e^{-x} + C_2(x)e^{3x}. \tag{44}$$

We find the derivatives of the functions $C_1(x)$ and $C_2(x)$ from the system of equations:

$$\begin{cases} C_1'(x)e^{-x} + C_2'(x)e^{3x} = 0, \\ -C_1'(x)e^{-x} + 3C_2'(x)e^{3x} = \dfrac{e^{3x}}{9+e^{2x}}. \end{cases}$$

To solve the system, we apply Cramer's rule:

$$\Delta = W(x) = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 4e^{2x};$$

$$\Delta_1 = \begin{vmatrix} 0 & e^{3x} \\ \dfrac{e^{3x}}{9+e^{2x}} & 3e^{3x} \end{vmatrix} = -\frac{e^{6x}}{9+e^{2x}};$$

$$\Delta_2 = W(x) = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \dfrac{e^{3x}}{9+e^{2x}} \end{vmatrix} = \frac{e^{2x}}{9+e^{2x}}.$$

Then, we have:

$$C_1'(x) = \frac{\Delta_1}{\Delta} = -\frac{1}{4} \cdot \frac{e^{4x}}{e^{2x}+9},$$

$$C_2'(x) = \frac{\Delta_2}{\Delta} = \frac{1}{4} \cdot \frac{1}{e^{2x}+9}.$$

By integrating the obtained equations, we get:

$$C_1(x) = -\frac{1}{4}\int \frac{e^{4x}}{e^{2x}+9}dx = \begin{vmatrix} u = e^{2x} \\ du = 2e^{2x}dx \end{vmatrix} = -\frac{1}{8}\int \frac{u^2 du}{u(u+9)} = \frac{1}{8}\int \frac{u}{u+9}du =$$

$$= -\frac{1}{8}\left[u - 9\ln|u+9|\right] + \hat{C}_1 = -\frac{1}{8}\left[e^{2x} - 9\ln\left(e^{2x}+9\right)\right] + \hat{C}_1,$$

$$C_2(x) = \frac{1}{4}\int \frac{1}{e^{2x}+9}dx = \frac{1}{4}\int \frac{1}{e^{2x}+9}dx = \begin{vmatrix} u = e^{2x} \\ du = 2e^{2x}dx \end{vmatrix} = \frac{1}{8}\int \frac{du}{u(u+9)} =$$

$$= \frac{1}{72}(\ln u - \ln|u+9|) + \hat{C}_2 = \frac{1}{36}\left[x - \frac{1}{2}\ln(e^{2x}+9)\right] + \hat{C}_2.$$

Notice that to find the functions $C_1(x)$ and $C_2(x)$, we could also directly use formulae (43):

$$C_1(x) = -\int \frac{y_2(x) f(x)}{W(x)} dx = -\frac{1}{4}\int \frac{e^{4x}}{e^{2x}+9} dx;$$

$$C_2(x) = \int \frac{y_1(x) f(x)}{W(x)} dx = \frac{1}{4}\int \frac{1}{e^{2x}+9} dx.$$

Substituting $C_1(x)$ and $C_2(x)$ in (44), we obtain a particular solution of the inhomogeneous equation:

$$\tilde{y}(x) = -\frac{e^{-x}}{8}\left[e^{2x} - 9\ln\left(e^{2x}+9\right)\right] + \frac{e^{3x}}{36}\left[x - \frac{1}{2}\ln(e^{2x}+9)\right].$$

Finally, the general solution of the given inhomogeneous linear differential equation, according to (37), is equal to $y(x) = y_o(x) + \tilde{y}(x)$, i. e., has the form:

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{e^{-x}}{8}\left[e^{2x} - 9\ln\left(e^{2x}+9\right)\right] + \frac{e^{3x}}{36}\left[x - \frac{1}{2}\ln(e^{2x}+9)\right].$$

Answer:

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{e^{-x}}{8}\left[e^{2x} - 9\ln\left(e^{2x}+9\right)\right] + \frac{e^{3x}}{36}\left[x - \frac{1}{2}\ln(e^{2x}+9)\right].$$

11 Systems of linear differential equations with constant coefficients

In this section, we will study systems of two first-order linear differential equations with constant coefficients.

Definition 28. An *explicit linear system of two first-order linear differential equations with constant coefficients* for functions $x(t)$ i $y(t)$ is called:

$$\begin{cases} x' = a_{11}x + a_{12}y + f_1(t) \\ y' = a_{21}x + a_{22}y + f_2(t) \end{cases}, \tag{40}$$

where $a_{ij} \in \mathbf{R}$ are coefficients for unknowns; $f_i = f_i(t)$, $i, j = 1,2$ are free terms of the equation.

Definition 29. If $f_i \equiv 0$, $i = 1,2$ the system of ODEs is called *homogeneous*, otherwise it is *inhomogeneous*.

Definition 30. A set of functions $\{\varphi(t), \psi(t)\}$ is called a *solution of the system* in the interval (a,b), if substituting them into the system equations instead of $x(t)$ and $y(t)$ transforms both equations into identities.

Definition 31. A *general solution of the system of two differential equations* (40) is a set of two differentiable functions of an independent variable t, which contain two arbitrary constants C_1, C_2:

$$\begin{aligned} x &= x(t, C_1, C_2), \\ y &= y(t, C_1, C_2) \end{aligned} \tag{41}$$

such that:

1) they satisfy the equations of system (40) for any real values of parameters C_1, C_2;

2) for given initial conditions: $x(t_0) = x_0$, $y(t_0) = y_0$ there will always be constant values C_1, C_2 that determine the solution of the system (40), which satisfies the initial conditions (41).

Definition 32. The solution of the system, obtained from the general solution at fixed values C_1, C_2 of parameters, is called a *particular solution* of the system of ODEs.

The *Cauchy problem* for a system of linear differential equations means finding a particular solution of the system (40) that satisfies the initial conditions (41).

We will consider two methods for solving a linear system of two first-order ODEs with constant coefficients.

Exclusion method. The essence of this method is to reduce the system of two first-order differential equations to a single second-order differential equation for one of the unknown functions.

First, we choose the function to be excluded. Here is an algorithm for solving system (40) by exclusion $x(t)$. For simplicity but without loss of generality consider the case of a homogeneous system

$$\begin{cases} x' = a_{11}x + a_{12}y, \\ y' = a_{21}x + a_{22}y. \end{cases} \tag{42}$$

Differentiate the second equation (42) to obtain:

$$y'' = a_{21}x' + a_{22}y'.$$

Now we substitute the expression for x' from the first equation (42):

$$y'' = a_{21}\left(a_{11}x + a_{12}y\right) + a_{22}y'. \tag{43}$$

From the second equation (42) we find:

$$x = \frac{1}{a_{21}}\left(y' - a_{22}y\right) \tag{44}$$

and substitute it into (43):

$$y'' = a_{11}y' - a_{11}a_{22}y + a_{12}a_{21}y + a_{22}y'.$$

So, we now have a homogeneous linear second-order differential equation with constant coefficients:

$$y'' - (a_{11} + a_{22})y' + (a_{11}a_{22} - a_{12}a_{21})y = 0.$$

We solve this equation, and then, using formula (44), we obtain the second unknown solution function $x(t)$ of the system, with $y(t)$ already found.

Remark 5. The exclusion method can also be applied to inhomogeneous systems of linear differential equations with constant coefficients. Such systems are reduced to inhomogeneous linear second-order differential equations.

Example 11. Let us solve the Cauchy problem for homogeneous system of two linear differential equations:

$$\begin{cases} x' = 2x + 3y \\ y' = 5x + 4y \end{cases}, \quad x(0) = 0, y(0) = 1. \tag{45}$$

Solution.

To obtain a differential equation for one of the unknown functions, we differentiate one of the system's equations. For example, calculate the second derivative of the function $y = y(t)$ from the second equation:

$$y'' = 5x' + 4y'.$$

Next, substitute the expression for the derivative $x'(t)$ from the first equation of (45) into the last equation:

$$y'' = 5(2x + 3y) + 4y'. \qquad (46)$$

Now we express the function $x(t)$ from the second equation of (45)

$$x = \frac{1}{5}(y' - 4y)) \qquad (47)$$

and substitute into (46):

$$y'' = 5 \cdot 2 \cdot \frac{1}{5}\,(y' - 4y) + 15y + 4y' = 6y' + 7y.$$

Thus, we obtain homogeneous second-order linear differential equation with constant coefficients:

$$y'' - 6y' - 7y = 0.$$

First, we find its general solution. For this compose a corresponding characteristic equation:

$$k^2 - 6k - 7 = 0.$$

The solutions of the last equation are real and distinct: $k_1 = -1, k_2 = 7$, so, according to Table 1, we get

$$y\,(t) = C_1 e^{-t} + C_2 e^{7t}.$$

From (47), we find the second unknown function:

$$x(t)=\frac{1}{5}(y'-4y)=\frac{1}{5}(-C_1e^{-t}+7C_2e^{7t}-4C_1e^{-t}-4C_2e^{7t})=$$

$$=-C_1e^{-t}+\frac{3}{5}C_2e^{7t}.$$

As a result, we obtain a general solution of the original system:

$$\begin{aligned} x(t)&=-C_1e^{-t}+\frac{3}{5}C_2e^{7t};\\ y(t)&=C_1e^{-t}+C_2e^{7t}. \end{aligned} \tag{48}$$

To solve the Cauchy problem, we equate $x(0)=0$, $y(0)=1$ and obtain the system for C_1, C_2:

$$\begin{cases} -C_1+\frac{3}{5}C_2=0;\\ C_1+C_2=1. \end{cases}$$

The solution $C_1=\frac{3}{8}, C_2=\frac{5}{8}$ of this system we substituted into the expressions $x(t)$, $y(t)$ and obtain the solution of the given Cauchy problem.

Answer: $\begin{cases} x(t)=-\frac{3}{8}\left(e^{-t}-e^{7t}\right);\\ y(t)=\frac{1}{8}\left(e^{-t}+5e^{7t}\right). \end{cases}$

Euler's method for solving a homogeneous system

Definition 33. The *characteristic equation* of a homogeneous system of linear differential equations with constant coefficients

$$\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \tag{49}$$

is the quadratic equation with respect to the parameter k defined as follows:

$$\begin{vmatrix} a_{11} - k & a_{12} \\ a_{21} & a_{22} - k \end{vmatrix} = 0. \tag{50}$$

Similar to the solution of a linear homogeneous equation with constant coefficients, the roots of the characteristic equation (50) define one of the functions of the general solution, which is the form of the second function.

Table 3 shows the correspondence between the roots of the characteristic equation and the general solution of a linear system.

Table 3 – General solution of a linear homogeneous system

№	Roots of the characteristic equation $k^2 + pk + q = 0$	General solution of the equation $\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$
1	$k_1, k_2 \in \mathbf{R}; \quad k_1 \neq k_2$	$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$, $C_1, C_2 \in \mathbf{R}$, $y = \frac{1}{a_{12}}\left(x' - a_{11}x\right)$
2	$k_1 = k_2 = k \in \mathbf{R}$	$x = \left(C_1 + C_2 t\right) e^{kt}$, $C_1, C_2 \in \mathbf{R}$, $y = \frac{1}{a_{12}}\left(x' - a_{11}x\right)$
3	$k_1, k_2 \in \mathbf{C}; \quad k_{1,2} = \alpha \pm i\beta$	$x = e^{\alpha t}\left(C_1 \cos\beta t + C_2 \sin\beta t\right)$, $C_1, C_2 \in \mathbf{R}$, $y = \frac{1}{a_{12}}\left(x' - a_{11}x\right)$

Example 12. Solve a homogeneous system of two linear differential equations using the Euler method:

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases}$$

Solution.

Form the characteristic equation (47) and solve it:

$$\begin{vmatrix} 2-k & 1 \\ 3 & 4-k \end{vmatrix} = 0;$$

$$(2-k)(4-k) - 3 = 0;$$

$$k^2 - 6k + 5 = 0;$$

$$k_1 = 1, \; k_2 = 5.$$

With the help of Table 3, we find a general solution $\{x(t), y(t)\}$ of the given homogeneous linear system:

$$x = C_1 e^t + C_2 e^{5t},$$

$$y = \dot{x} - 2x = C_1 e^t + 5C_2 e^{5t} - 2C_1 e^t - 2C_2 e^{5t} = -C_1 e^t + 3C_2 e^{5t}.$$

Answer: $x = C_1 e^t + C_2 e^{5t}$, $y = -C_1 e^t + 3C_2 e^{5t}$.

Theoretical questions for self-assessment

1 Give the definition of the differential equation of the n-th order and give its symbolic notation.

2 What is the solution of the n-th order differential equation?

3 What is the general solution of the n-th order differential equation?

4 Which solution of the equation is called a particular solution?

5 What types of first-order differential equations are you familiar with?

6 What is an explicit first-order differential equation?

7 Formulate the Cauchy problem for a first-order differential equation and provide its geometric interpretation.

8 What is the form of a differential equation with separable variables?

9 Which differential equation is called homogeneous? Describe a method to solve it.

10 Which first-order equation is called linear? Provide some examples.

11 Describe the solution of a first-order linear differential equation using the Bernoulli method.

12 What is the method of variation of an arbitrary constant?

13 What are called: "solution", "general solution", "particular solution", "general integral", "particular integral" of a second-order differential equation?

14 What is meant by the Cauchy problem for a second-order differential equation?

15 Which second-order differential equations allow a reduction of order?

16 What do you know about the properties of solutions of a second-order linear differential equation with constant coefficients?

17 What is the form of a homogeneous linear second-order differential equation with constant coefficients?

18 Which functions are called linearly independent, and which are linearly dependent on some interval?

19 What is the Wronskian? What properties of the Wronskian of solutions of a homogeneous linear second order differential equation do you know?

20 What is the structure of the general solution of a homogeneous linear second-order differential equation?

21 What is the characteristic equation for an inhomogeneous linear second-order differential equation with constant coefficients?

22 How to find the form of the particular solution of an inhomogeneous linear second-order equation with a special right-hand side?

23 What methods do you know for solving inhomogeneous linear second-order differential equations?

24 What is a homogeneous (inhomogeneous) system of linear differential equations?

25 What are the 'solution', 'general solution', 'particular solution', 'initial conditions', 'Cauchy problem' of a system of linear differential equations?

26 Describe the method of exclusion for solving systems of linear differential equations.

27 What is Euler's method for solving homogeneous systems of linear differential equations?

Test for self-assessment

1 Choose the first-order differential equations:

a) $y' + \frac{2xy}{1+x^2} = 2x^3$; b) $y'' = x^2 + 2^x$;

c) $y'' - 3y' = 0$; d) $y' = e^{3x} \sin^2 y$.

2 Choose the differential equations in differentials:

a) $xy^2 y' = y^3 + x^3$; b) $xydy = (x^2 - 1)dx$;

c) $ydx - 4xdy = 0$; d) $y' = x - 1$.

3 The general solution of the first-order order differential equation is:

a) any function $y = \varphi(x)$;

b) a function $y = \varphi(x)$ that turns the equation into identity;

c) a family of functions $y = \varphi(x, C)$ that are solutions for $\forall C \in \mathbf{R}$;

d) a family of functions $y = \varphi(x, C_1, C_2)$ that are solutions for $\forall C_i \in \mathbf{R}, i = 1, 2$.

4 The Cauchy problem for a first-order differential equation has the form:

a) $\begin{cases} y' = f(x, y), \\ y(0) = 0; \end{cases}$ b) $\begin{cases} y' = f(x, y), \\ y(x_0) = y_0; \end{cases}$

c) $\begin{cases} y' = f(x, y), \\ y(x_0) = y_0, \\ y'(x_0) = y_1; \end{cases}$ d) $\begin{cases} y' = f(x, y), \\ y(x_0) = y_0, \\ y'(x_1) = y_1. \end{cases}$

5 How many integral curves of the equation $y' = xy$ pass through the point (2;1) in the plane?

a) none; b) only one;

c) two; d) infinitely many.

6 The homogeneous first-order differential equation has the form:

a) $y' = f(x)$; b) $y' = f(x)g(y)$;

c) $y' + p(x)y = f(x)$; d) $y' = f\left(\dfrac{y}{x}\right)$.

7 Determine the type of the equation $y' + \dfrac{2xy}{1 + x^2} = 2x + 2x^3$:

a) homogeneous first-order ODE;

b) linear first-order ODE;

c) linear second-order ODE;

d) ODE with separable variables.

8 Choose the separable differential equations:

a) $y'y = x^2$; b) $y' = e^{3x} \sin^2 y$;

c) $y' - \frac{y}{x} = 2x^3$; d) $y'' - 3y' = 0$.

9 Choose the homogeneous first-order differential equations:

a) $y' - 2y = 0$; b) $x^2 y' + y^2 - xy = 0$;

c) $y' - xy^3 = 0$; d) $xy' = y \ln\left(\frac{y}{x}\right) + y$.

10 To solve a homogeneous first-order differential equation, one should make the substitution:

a) $z = y'$; b) $y = ux$;

c) $y = uv$; d) $y = x$.

11 To solve a linear first-order differential equation, one should make the change:

a) $z = y'$; b) $y = ux$;

c) $y = uv$; d) $y = x$.

12 Choose the linear first-order differential equations:

a) $2xy' = 2y + \frac{x^2}{y}$; b) $y' = \frac{x^2}{y}$;

c) $y' = e^{3x-y}$; d) $y' + 2xy = x^3 e^{-x^2}$.

13 Determine the type of the differential equation $y' + 2xy = xy^3$:

a) Bernoulli equation; b) first-order linear equation;

c) first-order homogeneous equation; d) ODE with separable variables.

14 What is the general solution of the equation $y' = \sqrt{x}$?

a) $y = \frac{1}{2\sqrt{x}} + C$; b) $y = 2\sqrt{x} + C$;

c) $y = \frac{3}{2}\sqrt[3]{x^2} + C$; d) $y = \frac{2}{3}\sqrt{x^3} + C$.

15 What is the general solution of the equation $y' = \sin 3x$?

a) $y = 3\cos 3x + C$; b) $y = -3\cos 3x + C$;

c) $y=\frac{1}{3}\cos 3x+C$; d) $y=-\frac{1}{3}\cos 3x+C$.

16 Choose the explicit differential equations:

a) $2xy'=2y+\frac{x^2}{y}$; b) $y'=2^{3x-y}$;

c) $y''+6y'-7y=0$; d) $y''=e^{-x}-x$.

17 Which of the given Cauchy problems are posed incorrectly?

a) $\begin{cases} y''=xy, \\ y(1)=3, \\ y'(1)=2; \end{cases}$ b) $\begin{cases} y'=x\sin^2 y, \\ y(1)=\pi/4, \\ y'(1)=1/2; \end{cases}$

c) $\begin{cases} y'=xy, \\ y(2)=1; \end{cases}$ d) $\begin{cases} y''-y'+12y=x, \\ y(0)=5; \end{cases}$

18 The Cauchy problem for second-order differential equations has the form:

a) $\begin{cases} y''=f(x,y,y'), \\ y(0)=0, \\ y'(0)=0; \end{cases}$ b) $\begin{cases} y''=f(x,y,y'), \\ y(x_0)=y_0; \end{cases}$

c) $\begin{cases} y''=f(x,y,y'), \\ y(x_0)=y_{0,} \\ y'(x_0)=y_1; \end{cases}$ d) $\begin{cases} y''=f(x,y,y'), \\ y(x_0)=y_{0,} \\ y'(x_1)=y_1. \end{cases}$

19 Which of the differential equations are of the second order?

a) $y''-3y'=0$; b) $2y'y=x^2$;

c) $y'=e^{x-y}$; d) $y''=xy$.

20 The general solution of a second-order differential equation is:

a) any function $y=\varphi(x)$;

b) a function $y=\varphi(x)$ that turns the equation into identity;

c) a family of functions $y=\varphi(x,C)$ that are solutions for $\forall C\in \mathbf{R}$;

d) a family of functions $y = \varphi(x, C_1, C_2)$ that are solutions for $\forall C_i \in \mathbf{R}, i = 1, 2$.

21 Choose the linear second-order differential equations:

a) $yy'' + (y')^2 + (y')^3 = 0$; b) $y'' - y = x^2$;

c) $xy' - y = -x^2$; d) $y'' - 3y' = 0$.

22 What is the general solution of the equation $y'' = \cos x$?

a) $y = \cos x + C_1 x + C_2$; b) $y = \sin x + C_1 x + C_2$;

c) $y = -\cos x + C_1 x + C_2$; d) $y = -\sin x + C_1 x + C_2$.

23 How to solve an incomplete second-order differential equation $F(x, y', y'') = 0$?

a) integrate the equation twice;

b) enter a new unknown function $u(x) = y'$;

c) enter a new unknown function $u(y) = y'$;

d) enter a new unknown function $u(x) = \dfrac{y}{x}$.

24 Choose the linear inhomogeneous second-order differential equations:

a) $y'' = y'$; b) $y'' + 8y' = 3x - 1$;

c) $y'' - y = \dfrac{e^x}{e^{2x} + 1}$; d) $y'' - (y')^2 = 0$.

25 The characteristic equation cannot have roots:

a) $k_1 = 3; k_2 = -3$; b) $k_1 = 5; k_2 = -3$;

c) $k_1 = 3i; k_2 = -3i$; d) $k_1 = 5i; k_2 = -3i$.

26 Find the roots of the characteristic equation of the differential equation $y'' - 8y' - 9y = 0$:

a) $k_1 = 9; k_2 = 1$; b) $k_1 = 9; k_2 = -1$;

c) $k_1 = -9; k_2 = 1$; d) $k_1 = -9; k_2 = -1$.

27 Find the form of the particular solution of a linear inhomogeneous second-order differential equation if $f(x) = xe^{2x}$, $k_{1,2} = \pm 2$:

a) $y = Ae^{2x}$; b) $y = Axe^{2x}$;

c) $y = (Ax + B)e^{2x}$; d) $y = x(Ax + B)e^{2x}$.

28 A particular solution of the equation $y'' - 3y' = (8x - 3)e^{4x}$, according to the method of undetermined coefficients should be sought in the form:

a) $y = C_1 + C_2 e^{3x}$; b) $y = (Ax + B)e^{4x}$;

c) $y = x(Ax + B)e^{4x}$; d) $y = Ae^{4x}$.

29 The characteristic equation for a system of differential equations

$$\begin{cases} x' = -3x + 4y \\ y' = 2x - y \end{cases}$$

has the form:

a) $\begin{vmatrix} -3k & 4 \\ 2 & -k \end{vmatrix} = 0$; b) $\begin{vmatrix} -3-k & 4 \\ 2 & -1-k \end{vmatrix} = 0$;

c) $\begin{vmatrix} -3 & 4k \\ 2k & -1 \end{vmatrix} = 0$; d) $\begin{vmatrix} -3 & 4-k \\ 2-k & -1 \end{vmatrix} = 0$.

30 The characteristic equation for a system of differential equations

$$\begin{cases} x' = x + 4y \\ y' = 2x + 3y \end{cases}$$

has the form:

a) $k^2 - 3k + 4 = 0$; b) $k^2 - 4k + 3 = 0$;

c) $k^2 - 4k - 5 = 0$; d) $k^2 - 5k + 4 = 0$.

Individual tasks

Task 1. Solve the first-order differential equations.

1	a) $y'y^2 - y\sin x = 0$	b) $(2x - y)y' = x + 2y$
2	a) $(x-1)y' = x^3 y$	b) $xy' = y\ln\left(\frac{y}{x}\right) + y$
3	a) $xy' - \left(x^3 - 1\right)e^y = 0$	b) $xy' - y = 3\sqrt{y^2 + x^2}$
4	a) $\left(xy + x^3 y\right)y' = 1 + y^2$	b) 16.$xy' = 2\left(y - \sqrt{xy}\right)$
5	a) $2xyy' - \ln x = 0$	b) $x^2 y' = y^2 + 6xy + 6x^2$
6	a) $x^4 y' - x\sin^2 y = 0$	b) $\left(x^2 - 6xy\right)y' = x^2 - 5y^2 + xy$
7	a) $\left(x^2 - 3\right)y' = xtgy$	b) $xy'\left(7x^2 + 2y^2\right) = 3y^3 + 14yx^2$
8	a) $x^2 y' = 1 - \sqrt[3]{y}$	b) $y' = (x + y)^2$
9	a) $y' + y^3 ctgx = 0$	b) $xy' = 4\sqrt{2x^2 + y^2}$
10	a) $y'\sqrt{2y-1} - y5^x = 0$	b) $3x^2 y' - y^2 = 10xy + 10y^2$
11	a) $x^2 y'\ln y = y - xy$	b) $x^2 y' = y^2 + 4xy + 2x^2$
12	a) $xy^2 y' = 2 + \log_2 x$	b) $4x^2 y' - y^2 = 10xy - 5x^2$
13	a) $xyy' = x^2 - 1$	b) $xy' - y - \sqrt{x^2 + y^2} = 0$
14	a) $y'(xy - x) - y^2 = 0$	b) $xy' - y + x\sin\frac{y}{x} = 0$
15	a) $x^2 y' - \left(\sqrt[5]{x} - 1\right)y^2 = 0$	b) $y'\cos\frac{y}{x} = \frac{y}{x}\cos\frac{y}{x} - 1$
16	a) $y' - xy\sqrt{x^2 - 3} = 0$	b) $y' = (x + 3y)^2$
17	a) $xy^3 y' = 1 - \lg_2 x$	b) $2x^2 y' = y^2 + 6xy + 3x^2$

18	a) $y'\sqrt{5-y}-xye^{x}=0$	b) $y'\left(x^2-2xy\right)=x^2-y^2+2xy$
19	a) $x^3y'-x\cos^2 y=0$	b) $3x^2y'=y^2+8xy-4x^2$
20	a) $y'-xy\sqrt{2-x^2}=0$	b) $y'\sin\frac{y}{x}=\frac{y}{x}\sin\frac{y}{x}-1$
21	a) $y'\sqrt{y}-y\cos 3x=0$	b) $2xy'=2y+\frac{x^2}{y}$
22	a) $y'y-x\sqrt{y^2+5}=0$	b) $xy'=y+x\sin^2\frac{y}{x}$
23	a) $xy'\sqrt{y}=\ln^3 x+\sqrt{x}$	b) $3xy'=3y+x\cos^2\frac{y}{x}$
24	a) $y'\sqrt{x}-x\sin^2 y=0$	b) $y'\left(xy-x^2\right)-y^2=0$
25	a) $y'y+y^3\sin 2x=0$	b) $xy'+2\sqrt{xy}-y=0$
26	a) $y'\sqrt{x^3}-2x\cos^2 y=0$	b) $\left(x^2-6xy\right)y'=x^2-5y^2+xy$
27	a) $xyy'=3\ln^2 x-x^4$	b) $\left(x^2-y^2\right)y'=2xy$
28	a) $y'y^2-y\cos^2 x=0$	b) $x^2y'=y^2-x^2$
29	a) $y'y-e^{x}\sqrt{3y+1}=0$	b) $3x^2y'-3y^2=x^2$
30	a) $\left(x+5\right)y'+x^2y-y=0$	b) $x^2y'=xy+2y^2$

Task 2. Solve the Cauchy problem for a linear first-order differential equation using two methods and check the correctness by substitution.

1	$y'\sin x+y\cos x=1;\quad y\left(\frac{\pi}{2}\right)=1$
2	$y'-\frac{y}{x+1}=e^{x}(x+1);\quad y(0)=1$

3	$y' + y\cos x = \frac{\sin 2x}{2}; \quad y(0) = 0$
4	$y' + y\text{tg}x = \cos^2 x; \quad y\left(\frac{\pi}{4}\right) = \frac{1}{2}$
5	$y' - \frac{y}{x+2} = x^2 + 2x; \quad y(-1) = \frac{3}{2}$
6	$y' - \frac{y}{x} = x^2; \quad y(1) = 0$
7	$y' - 4yx = x; \quad y(0) = \frac{3}{4}$
8	$y' - \frac{y}{x} = x\sin x; \quad y\left(\frac{\pi}{2}\right) = 1$
9	$y' + \frac{2xy}{1+x^2} = 2x + 2x^3; \quad y(0) = \frac{2}{3}$
10	$y' - y\text{ctg}x = 2x\sin x; \quad y\left(\frac{\pi}{2}\right) = 0$
11	$y' + \frac{y}{x} = \frac{e^x(x+1)}{x}; \quad y(1) = 0$
12	$y' + 2xy = 2xe^{-x^2}; \quad y(1) = 5$
13	$y' - \frac{2xy}{1+x^2} = 1 + x^2; \quad y(-2) = 5$
14	$y' - y\sin x = e^{-\cos x}\sin 2x; \quad y\left(\frac{\pi}{2}\right) = 3$
15	$y' + y = \frac{e^{-x}}{1+x^2}; \quad y(0) = 2$
16	$y' + \frac{y}{2x} = x^3; \quad y(1) = 1$
17	$xy' - 3y = x^4e^x; \quad y(1) = e$
18	$xy' + 2y = \frac{1}{x}; \quad y(3) = 1$

19	$xy' + y = \frac{2x}{1+x^2}; \quad y(1) = 0$
20	$y'\cos x - 2y\sin x = 2; \quad y(0) = 3$
21	$xy' + y = (1+x)e^x; \quad y(1) = e$
22	$y' - y\cos x = x^2 e^{\sin x}; \quad y(0) = 2$
23	$(1+x^2)y' - 2xy = x + \frac{1}{x}; \quad y(1) = 1$
24	$y' - 4y\cos^2 x = (1+x^2)e^{\sin 2x}; \quad y(0) = 2$
25	$y' + 2y = e^{-2x}\mathrm{tg}x; \quad y(0) = 2$
26	$y' + y\cos x = \cos x; \quad y(0) = 4$
27	$y'(1+x) - 2y = (1+x)^3 e^x; \quad y(0) = 1$
28	$x^2 y' + (1-2x)y = x^2; \quad y(1) = 1$
29	$xy' + y = x^2 + 3x + 2; \quad y(1) = 2$
30	$y' + 2xy = 2x; \quad y(0) = 2$

Task 3. Find a general solution or a general integral of the second-order differential equation using reduction of order.

1	$xy'' = 2y'$	16	$y'' - 3\frac{(y')^2}{y} = 8y'$
2	$y'' + y'\,\mathrm{tg}x = \sin 2x$	17	$(x+1)y'' + y' = 0$
3	$1 + (y')^2 - yy'' = 0$	18	$y^3 y'' + y' = 0$
4	$xy'' + 2y' = x^3$	19	$xy'' = y'\ln\frac{y'}{x}$
5	$yy'' + (y')^2 + (y')^3 = 0$	20	$yy'' - 2y^2 y' - (y')^2 = 0$

6	$y'' + y'\frac{1}{x} = x^2$	21	$y'' = e^{2y}$
7	$y''(1+y) - 5(y')^2 = 0$	22	$(y-1)y'' = 2(y')^2$
8	$y''\,\mathrm{tg}y = 2(y')^2$	23	$xy'' + y' - 2x - 3 = 0$
9	$y'' - 2y'tgx = \sin x$	24	$yy'' + (y')^2 = y'y$
10	$3yy'' + (y')^2 = 0$	25	$xy''y' - (y')^2 = x^4$
11	$y'' - 2\frac{(y')^2}{y} + y' = 0$	26	$y''(1+y^2) = 2y(y')^2$
12	$xy'' + y' = x + 1$	27	$(1+x^2)y'' + (y')^2 + 1 = 0$
13	$y'' + (y')^2\,ctgy = 0$	28	$8yy'' = 1 + 4(y')^2$
14	$y''y = 3(y')^2$	29	$y'' = \sqrt{1-(y')^2}$
15	$(1-x^2)y'' = xy'$	30	$y'' = (y')^3 e^y.$

Task 4. Solve the Cauchy problem for a linear second-order differential equation using two methods and verify the correctness of the solution.

1	$y'' + 4y' + 4y = xe^{3x}; \quad y(0) = -1; \quad y'(0) = 1$
2	$y'' - 5y' + 6y = 3 - x^2; \quad y(0) = 5; \quad y'(0) = 6$
3	$y'' + y = x\cos 2x; \quad y(0) = -1; \quad y'(0) = 1$
4	$y'' + 2y' + y = xe^x; \quad y(0) = 0; \quad y'(0) = -2$
5	$y'' + 4y = 4x^2 - x; \quad y(0) = 0; \quad y'(0) = 3$
6	$y'' + 9y = 3\sin 3x; \quad y(0) = 0; \quad y'(0) = 2$
7	$y'' - 2y' - 3y = (x+5)e^{2x}; \quad y(0) = 0; \quad y'(0) = 2$
8	$y'' + 4y' = e^{-x}(x-9); \quad y(0) = 1; \quad y'(0) = 0$

9	$y'' + 16y = x\sin 2x; \quad y(0) = 0; \quad y'(0) = 1$
10	$y'' + 8y' + 16y = xe^{-4x}; \quad y(0) = 0; \quad y'(0) = 2$
11	$y'' - 4y' + 5y = 8x^2 - 3; \quad y(0) = 2; \quad y'(0) = 1$
12	$y'' + 4y = 3x\sin x; \quad y(0) = 0; \quad y'(0) = -1$
13	$y'' + 6y' + 9y = 2xe^{-3x}; \quad y(0) = 6; \quad y'(0) = 0$
14	$y'' - 2y' = 2x^2 - x + 7; \quad y(0) = 1; \quad y'(0) = -1$
15	$y'' + 2y' + 5y = -3\sin 2x; \quad y(0) = 1; \quad y'(0) = -2$
16	$y'' - 4y' + 4y = e^{2x}(2x - 3); \quad y(0) = -1; \quad y'(0) = 2$
17	$y'' + 6y' + 9y = 2\cos 2x + 3\sin 2x; \quad y(0) = 2; \quad y'(0) = 1$
18	$y'' - 4y' + 8y = xe^{4x}; \quad y(0) = -2; \quad y'(0) = 1$
19	$y'' - 3y' + 2y = 2x^2 + 1; \quad y(0) = 3; \quad y'(0) = -1$
20	$y'' + 6y' + 13y = e^x \cos 4x; \quad y(0) = 0; \quad y'(0) = 1$
21	$y'' + y' - 6y = (3 - 4x)e^{2x}; \quad y(0) = 2; \quad y'(0) = -1$
22	$y'' - 2y' = 4x^2 + 9; \quad y(0) = 1; \quad y'(0) = 3$
23	$y'' + y' - 2y = \sin 2x + 3\cos 2x; \quad y(0) = 2; \quad y'(0) = -1$
24	$y'' - 6y' + 9y = xe^{3x}; \quad y(0) = -1; \quad y'(0) = 2$
25	$y'' + 4y' + 3y = 6x^2 - 1; \quad y(0) = 1; \quad y'(0) = 2$
26	$y'' - 8y' + 25y = 2\cos 3x; \quad y(0) = -2; \quad y'(0) = 2$
27	$y'' + 3y' + 2y = xe^{-x}; \quad y(0) = -2; \quad y'(0) = 0$
28	$y'' + 2y' - 3y = (3x + 2)^2; \quad y(0) = 3; \quad y'(0) = 0$
29	$y'' + y' - 12y = xe^{3x}; \quad y(0) = 4; \quad y'(0) = 0$
30	$y'' - 6y' - 7y = 2\sin 5x; \quad y(0) = 1; \quad y'(0) = 0$

Task 5. Solve the inhomogeneous linear second-order differential equation using the method of variation of arbitrary constants.

1	$y''+9y=\frac{\sin 6x}{2+\sin^2 3x}$	16	$y''+9y'+18y=\frac{e^{-2x}}{1+e^{2x}}$
2	$y''+y'-2y=\frac{e^{2x}}{3+e^{x}}$	17	$y''+4y'+3y=\frac{e^{-x}}{2+e^{x}}$
3	$y''-6y'+9y=\frac{xe^{3x}}{x^2+5x+4}$	18	$y''+16y=\frac{1}{3+\cos 8x}$
4	$y''-2y'-8y=\frac{e^{5x}}{1+e^{2x}}$	19	$y''-5y'+4y=\frac{e^{4x}}{9+e^{2x}}$
5	$y''-3y'+2y=\frac{e^{3x}}{4+e^{x}}$	20	$y''-10y'+25y=\frac{e^{5x}}{x^2+6x+8}$
6	$y''-6y'+8y=\frac{e^{4x}}{4+e^{2x}}$	21	$y''-4y'+3y=\frac{e^{3x}}{4+e^{x}}$
7	$y''-8y'+16y=\frac{xe^{4x}}{9+x^2}$	22	$y''+25y=\frac{1}{\cos^2 5x+8}$
8	$y''+7y'+12y=\frac{e^{-2x}}{1+e^{2x}}$	23	$y''+5y'+4y=\frac{e^{x}}{4+e^{2x}}$
9	$y''-9y'+20y=\frac{e^{6x}}{1+e^{2x}}$	24	$y''+4y'+4y=\frac{e^{-2x}}{x^2+8x+12}$
10	$y''+6y'+9y=\frac{e^{-3x}}{4+x^2}$	25	$y''-7y'+12y=\frac{e^{5x}}{5+e^{x}}$
11	$y''+9y'+20y=\frac{e^{-3x}}{4+e^{x}}$	26	$y''-y'-2y=\frac{e^{2x}}{5+e^{x}}$
12	$y''+8y'+16y=\frac{xe^{-4x}}{x^2+9}$	27	$y''-4y'+4y=\frac{xe^{2x}}{1+x^2}$

13	$y'' - 8y + 15y = \dfrac{e^{5x}}{4 + e^{2x}}$	28	$y'' + 4y = \dfrac{\sin 4x}{3 + \cos^2 2x}$
14	$y'' + 10y' + 25y = \dfrac{xe^{-5x}}{x^2 + 4x + 3}$	29	$y'' - 16y = \dfrac{1}{5 + \sin^2 4t}$
15	$y'' - 7y' = \dfrac{e^{3x}}{9 + e^{2x}}$	30	$y'' - 4y = \dfrac{\sin 4t}{5 + \cos^2 2t}$

Task 6. Find the general solution of a homogeneous linear system of first-oder differential equations using two methods.

1	$\begin{cases} x' = -2x + 3y \\ y' = x - 4y \end{cases}$	16	$\begin{cases} x' = -4x - 5y \\ y' = x + 2y \end{cases}$
2	$\begin{cases} x' = 5x - 3y \\ y' = 4x - 3y \end{cases}$	17	$\begin{cases} x' = x + 4y \\ y' = 2x + 3y \end{cases}$
3	$\begin{cases} x' = -3x + 4y \\ y' = 2x - y \end{cases}$	18	$\begin{cases} x' = x + 4y \\ y' = 2x + 3y \end{cases}$
4	$\begin{cases} x' = 2x + 7y \\ y' = x - 4y \end{cases}$	19	$\begin{cases} x' = 3x - 4y \\ y' = -2x + y \end{cases}$
5	$\begin{cases} x' = 4x - 7y \\ y' = 2x - 5y \end{cases}$	20	$\begin{cases} x' = -4x + 7y \\ y' = -2x + 5y \end{cases}$
6	$\begin{cases} x' = 4x + 7y \\ y' = -2x - 5y \end{cases}$	21	$\begin{cases} x' = 4x + 2y \\ y' = -7x - 5y \end{cases}$
7	$\begin{cases} x' = 5x + 3y \\ y' = -4x - 3y \end{cases}$	22	$\begin{cases} x' = -2x + 7y \\ y' = x + 4y \end{cases}$
8	$\begin{cases} x' = 2x + 3y \\ y' = x + 4y \end{cases}$	23	$\begin{cases} x' = -5x + 3y \\ y' = -4x + 3y \end{cases}$
9	$\begin{cases} x' = 2x + 7y \\ y' = x - 4y \end{cases}$	24	$\begin{cases} x' = -2x - 3y \\ y' = -x - y \end{cases}$

10	$\begin{cases} x' = 2x - 3y \\ y' = -x + 4y \end{cases}$	25	$\begin{cases} x' = 2x - y \\ y' = -7x - 4y \end{cases}$
11	$\begin{cases} x' = -5x - 3y \\ y' = 4x + 3y \end{cases}$	26	$\begin{cases} x' = 3x + 2y \\ y' = 4x + y \end{cases}$
12	$\begin{cases} x' = -2x - 7y \\ y' = -x + 4y \end{cases}$	27	$\begin{cases} x' = -4x - 7y \\ y' = 2x + 5y \end{cases}$
13	$\begin{cases} x' = 3x + 4y \\ y' = 2x + y \end{cases}$	28	$\begin{cases} x' = -3x - 4y \\ y' = -2x - y \end{cases}$
14	$\begin{cases} x' = -5x - 7y \\ y' = 2x + 4y \end{cases}$	29	$\begin{cases} x' = 2x + 3y \\ y' = 5x + 4y \end{cases}$
15	$\begin{cases} x' = -3x - 3y \\ y' = 4x + 5y \end{cases}$	30	$\begin{cases} x' = 2x + 3y \\ y' = 2x + 7y \end{cases}$

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Appendix

Table A.1 – The table of derivatives

Function $y = f(x)$	Derivative $y' = f'(x)$
$C = const$	0
$x^{\alpha},\ \alpha \in R$	$\alpha \cdot x^{\alpha-1}$
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\mathrm{tg}\, x$	$\frac{1}{\cos^2 x}$
$\mathrm{ctg}\, x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\mathrm{arctg}\, x$	$\frac{1}{1+x^2}$
$\mathrm{arcctg}\, x$	$-\frac{1}{1+x^2}$

Table A.2 – The table of basic integrals

1	$\int 0dx = C$
2	$\int dx = x + C$
3	$\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$
4	$\int \frac{dx}{x} = \int x^{-1}dx = \ln\|x\| + C$
5	$\int a^x dx = \frac{a^x}{\ln a} + C, \qquad a > 0, a \neq 1$
6	$\int e^x dx = e^x + C$
7	$\int \sin x dx = -\cos x + C$
8	$\int \cos x dx = \sin x + C$
9	$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$
10	$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$
11	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C, \quad a > 0$
12	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C, \quad a > 0$
13	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + C, \quad a \neq 0$
14	$\int \frac{dx}{\sqrt{x^2 + a}} = \ln\left\|x + \sqrt{x^2 + a}\right\| + C, \quad a \neq 0$
15	$\int \frac{dx}{x - a} = \ln\|x - a\| + C$
16	$\int \operatorname{tg} x dx = -\ln\|\cos x\| + C$

17	$\int \operatorname{ctg} x dx = \ln\lvert\sin x\rvert + C$
18	$\int \csc x dx = \int \frac{dx}{\sin x} = \ln\left\lvert\operatorname{tg}\frac{x}{2}\right\rvert + C$
19	$\int \sec x dx = \int \frac{dx}{\cos x} = \ln\left\lvert\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\rvert + C$

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